A note on the Dutch Book method*

J. B. Paris
Department of Mathematics
University of Manchester
Manchester M13 9PL
UK
jeff@ma.man.ac.uk

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Abstract

The paper considers generalizing the classical Dutch Book argument for identifying degree of belief with probability to yield the corresponding analogs of probability functions for various non-standard propositional logics for example modal, intuitionistic, and paraconsistent logics.

1 Introduction and Notation

The purpose of this modest note is to explicitly point out some consequences of applying the classic Dutch Book justification for rational belief being identified with probability in the context of alternate, non-Tarskian, semantics. Whilst the formal version of the Dutch Book Theorem that we shall need is well know from the work of De Finetti, see [5] p90, our observations concerning the consequences of this theorem for characterizing rational belief functions in the context of certain alternate, non-standard, semantics would, with the exception of a specific earlier example due to Jaffray, [8] (see also Regoli, [13]), appear not to be widely appreciated. Some related work on the many-valued Łukasiewicz logics $Ł_{n+1}$ (in this note we are primarily concerned with two valued semantics) has been published by Gerla in [7] and will be discussed briefly in the concluding section.

Before formally stating these results we need some notation. Let L be a finite propositional language and SL the set of sentences of L, say using the standard

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connectives $\vee, \wedge, \neg, \rightarrow$. As usual we use $p, q, ...$ etc for the propositional variables of L and $\theta, \phi, ...$ etc for elements of SL.

Let $\mathbb{V}$ be a subset of the set of all functions from SL into $\{0, 1\}$. We should think of $\mathbb{V}$ as the set of 'possible worlds', where for $V \in \mathbb{V}$ and $\theta \in SL$, θ is *true/false* in V if $V(\theta) = 1/0$. To start with we shall assume that $\mathbb{V}$ is finite.

Let $\mathbb{B}$ be the set of all functions from SL into $[0, 1]$. We are thinking of the elements of $\mathbb{B}$ as possible subjective *belief functions* on SL. In other words for $\theta \in SL$ and $B \in \mathbb{B}$ we should think of $B(\theta)$ as a measure of an agent's 'belief' (on the scale $[0, 1]$, 1 signifying maximum possible belief etc) to θ 'being true' in the *actual world*, V ($V \in \mathbb{V}$).

One particularly relevant example of such a set $\mathbb{V}$ is the set $\mathbb{V}^T$ of (classical) valuations on SL, that is functions $V : SL \longrightarrow \{0, 1\}$ satisfying the Tarski truth conditions that for $\theta, \phi \in SL$

(T1) $V(\neg\theta) = 1 \iff V(\theta) = 0,$
(T2) $V(\theta \wedge \phi) = 1 \iff V(\theta) = 1 \ \& \ V(\phi) = 1,$
(T3) $V(\theta \vee \phi) = 0 \iff V(\theta) = 0 \ \& \ V(\phi) = 0,$
(T4) $V(\theta \rightarrow \phi) = 0 \iff V(\theta) = 1 \ \& \ V(\phi) = 0.$

In this case $\mathbb{V}^T$ is finite since L is finite and every $V \in \mathbb{V}^T$ is determined by its values on the propositional variables alone.

Theorem 1 *[4],[5]*

The function $B \in \mathbb{B}$ is a probability function, that is satisfies[1] *that for all $\theta, \phi \in SL$,*

(P1) If $\models \theta$ then $B(\theta) = 1$,
if $\models \neg\theta$ then $B(\theta) = 0$
(P2) If $\theta \models \phi$ then $B(\theta) \leq B(\phi)$
(P3) $B(\theta \vee \phi) + B(\theta \wedge \phi) = B(\theta) + B(\phi)$

if and only if there does not exist a Dutch Book against B. That is, there do not exist some $\theta_1, \theta_2, ..., \theta_m \in SL$ and $s_1, s_2, ..., s_m \in \mathbb{R}$ such that for all $V \in \mathbb{V}^T$,

$$\sum_{i=1}^{m} s_i(V(\theta_i) - B(\theta_i)) < 0.$$

The relevance of the term 'Dutch Book' here is as follows. Suppose that we identify one's belief in θ, $B(\theta)$, with one's limiting willingness to bet that θ is true in the actual world V (V a member of the set $\mathbb{V}^T$ of *possible worlds*). In other words $B(\theta)$ is characterized by the properties that for stake $s > 0$ and $\eta < Bel(\theta)$ one is willing to pay out at least $s\eta$ to receive s if $V(\theta) = 1$ (and nothing if $V(\theta) = 0$) whilst for any $\zeta > Bel(\theta)$ one is willing to accept the 'reverse bet' in which one receives at least $s\zeta$ on penalty of having to pay out 1

[1] There is some redundancy in these conditions but they are presented in this way to facilitate comparison with subsequent definitions

if $V(\theta) = 1$ (and nothing if $V(\theta) = 0$)[2]. In this case the expression on the left of the inequality in this theorem is easily seen to be the greatest lower bound on one's profit (negative profit = loss) from simultaneously entering into such bets on the θ_i, $i = 1, 2, ..., r$, at stakes s_i. Thus having a Dutch Book against one means that for a suitably tight choice of η's and ζ's it would be possible arrange a finite set of bets, each of which was individually acceptable to one but whose combined effect would guarantee one a loss no matter what the true state, V, of the world was.

The upshot of this then is that if we identify one's belief with the limit of one's willingness to bet then one's beliefs can be rational, in the sense of avoiding any Dutch Book, if and only if they determine a probability function.

Theorem 1 applies to the classical situation where the semantics are given by Tarski's conditions (T1)-(T4) and 'worlds' are equated with valuations $V \in \mathbb{V}^T$. However these are certainly not the only sorts of 'worlds' which are possible. In non-standard logics it is customary to consider semantics for languages containing additional features, for example modalities, and where the connectives have alternate interpretations. Within the context of these semantics we can still consider beliefs as reflecting limiting willingness to bet on what holds in the 'true world'. Furthermore, just as in the classical case, avoidance of a Dutch Book (the *Dutch Book method* of our title) provides a criterion of rationality for judging such belief functions.

In the literature concerned purely with justifying probability as the rational quantification of belief, see for example the early sections of [4], [9], Theorem 1 is commonly proved directly in the right to left direction by showing that from any failure of (P1)-(P3) one can directly construct a Dutch Book. However splitting the argument up into two steps clarifies the role played by the underlying semantics.

The first step is just the following special case of the 'full Dutch Book Theorem', see [5], where a 'Dutch Book against B' is defined as in Theorem 1 but with the set of possible worlds $\mathbb{V}$ in place of $\mathbb{V}^T$. For the sake of completeness we include a proof.

Theorem 2 *For $\mathbb{V}$ finite a function $B \in \mathbb{B}$ does not permit a Dutch Book if and only if B is a convex combination of the functions in $\mathbb{V}$, i.e. $B = \sum_{V \in \mathbb{V}} a_V V$ where the $a_V \geq 0$ and $\sum_{V \in \mathbb{V}} a_V = 1$.*

[2]We should emphasize here that the existence of such a value $B(\theta)$ is predicated on the assumption, which we shall make throughout this paper, that for any $\theta \in SL$, stake $s > 0$ and $\eta \in [0, 1]$ *either* one is willing to accept a bet in which one pays out $s\eta$ to receive s from a bookmaker if θ turns out to be true (and nothing otherwise) *or* one is willing to reverse the roles and sell the bookmaker a bet in which s/he pays you $s\eta$ to receive s from you if θ turns out true (and nothing otherwise). [For full details of this argument within the current context see [10] p20.]

Proof. Suppose on the contrary that B was a convex combination $\sum_{V\in\mathbb{V}} a_V V$ of functions in $\mathbb{V}$ but that for some $\theta_i \in SL$ and $s_i \in \mathbb{R}$, $i = 1, ..., m$,

$$\sum_{i=1}^{m} s_i(V(\theta_i) - B(\theta_i)) < 0$$

for all $V \in \mathbb{V}$. In that case

$$\sum_{V\in\mathbb{V}} a_V \left\{ \sum_{i=1}^{m} s_i(V(\theta_i) - B(\theta_i)) \right\} < 0.$$

But changing the order of summation gives

$$\sum_{i=1}^{m} \left\{ s_i(\sum_{V\in\mathbb{V}} a_V V(\theta_i)) - (\sum_{V\in\mathbb{V}} a_V) s_i B(\theta_i) \right\} < 0,$$

i.e.

$$\sum_{i=1}^{m} \{ s_i B(\theta_i) - s_i B(\theta_i) \} < 0,$$

a contradiction.

The converse is a special case of a well known result from linear algebra but for the sake of completeness we shall give the details. So suppose that B is not a convex combination of elements from $\mathbb{V}$. Then we can find sentences $\theta_1, ..., \theta_m \in SL$ such that $B \restriction \{\theta_1, ..., \theta_m\}$ (i.e. B restricted to the set $\{\theta_1, ..., \theta_m\}$) is not a convex combination of the functions $V \restriction \{\theta_1, ..., \theta_m\}$ for $V \in \mathbb{V}$ (otherwise by taking a suitably convergent subsequence as the $\theta_1, ..., \theta_m$ extend through all of SL we could contradict our initial assumptions). This means that $< B(\theta_1), ..., B(\theta_m) >$ is not in the closed convex set Y of vectors of the form

$$\sum_{V\in\mathbb{V}} a_V < V(\theta_1), ..., V(\theta_m) >$$

where the $a_V \geq 0$, and $\sum_{V\in\mathbb{V}} a_V = 1$. Hence by the well known Separating Hyperplane Theorem for convex sets (see the appendix for a proof) there must be a vector $\vec{s} =< s_1, ..., s_m > \in \mathbb{R}^m$ such that

$$(< v_1, ..., v_m > - < B(\theta_1), ..., B(\theta_m) >) \cdot \vec{s} < 0$$

for all $< v_1, ..., v_m > \in Y$. In particular then substituting $< V(\theta_1), ..., V(\theta_m) >$ for $< v_1, ..., v_m >$ for V ranging over $\mathbb{V}$ shows that we have here a Dutch Book for B. ■

This result, of course, applies to any (at present) finite set of 'possible worlds', indeed the restriction to just the truth values 0 and 1 can clearly be relaxed, we imposed it simply to remain on reasonably familiar territory[3]. The standard

[3]Related to this see [7].

Dutch Book argument for belief as probability now follows if we take $\mathbb{V} = \mathbb{V}^T$. For then to avoid a Dutch Book B must be of the form $\sum_{V \in \mathbb{V}^T} a_V V$ with $\sum_{V \in \mathbb{V}^T} a_V = 1$, $a_V \geq 0$. But each $V \in \mathbb{V}^T$ is uniquely determined by the (unique) *atom* α_V, that is sentence of the form $\pm p_1 \wedge \pm p_2 \wedge ... \wedge \pm p_n$, such that $V(\alpha_V) = 1$. So $B(\alpha_V) = a_V$ and for $\theta \in SL$,

$$\begin{aligned} B(\theta) &= \sum_{V \in \mathbb{V}^T} a_V V(\theta) \\ &= \sum\{\alpha_V \mid V \in \mathbb{V}^T, \alpha_V \models \theta\}, \\ &\quad \text{since } V(\theta) = 1 \iff \alpha_V \models \theta, \\ &= \sum\{\ B(\alpha_V) \mid V \in \mathbb{V}^T, \alpha_V \models \theta\} \end{aligned} \tag{1}$$

and it is well known (see for example [10]) that every probability function on SL arises in this way, and conversely.

Were one only interested here in Theorem 1 this would certainly be a pretty long winded proof! The advantage of this derivation however is that we have in Theorem 2 a general result which can be applied to many other situations where our notion of 'truth' is not simply confined to be Tarskian, i.e. satisfying (T1-4). At the same time it highlights the dependence of belief as probability on the classical, Tarskian, interpretation of truth.

A second nice property of this proof is that it gives as a simple corollary a generalization of a theorem of Lehman, [9] (see also theorem 1 of [13]). The generalization concerns the possibility of forming a Dutch Book when the function B is only *partial* (so in the definition of a Dutch Book the $\theta_1, ..., \theta_m$ are also required to be in the domain of B). In the following theorem $\mathbb{V}$ is, as usual, a subset of the set of all functions from SL to $\{0, 1\}$.

Corollary 3 *Let B be a partial function from SL into $[0, 1]$. Then B does not permit a Dutch Book if and only if B has an extension to a function in $\mathbb{B}$ which does not permit a Dutch Book, equivalently has an extension to a convex combination of functions in $\mathbb{V}$.*

Proof. Clearly the implication follows from right to left by Theorem 2. In the other direction suppose B is partial and does not permit a Dutch Book. For $V \in \mathbb{V}$ let V^- be the restriction of V to the domain of B. Clearly B does not permit a Dutch Book with respect to the set of 'valuations' $\{V^- \mid V \in \mathbb{V}\}$ so by an immediate adaption of Theorem 2 B is a convex combination $\sum_{V \in \mathbb{V}} a_V V^-$. The required extension of B to a total function is $\sum_{V \in \mathbb{V}} a_V V$. ■

Up to now we have required $\mathbb{V}$ to be finite. For possibly infinite $\mathbb{V}$ theorem 2 becomes:

Corollary 4 *For $\mathbb{V}$ possibly infinite, a function $B \in \mathbb{B}$ does not permit a Dutch Book if and only if for every finite $\Gamma \subset SL$ $B \restriction \Gamma$ is a convex combination of the functions in $\{V \restriction \Gamma \mid V \in \mathbb{V}\}$.*

2 Applications to Non-Tarskian Truth

We now turn to considering some other applications of Theorem 2 to alternate notions of 'truth'. As we shall see the vital, and often missing, ingredient turns out to be an argument corresponding to (1) above for giving local characterizations of those $B \in \mathbb{B}$ which are convex combinations of the $V \in \mathbb{V}$.

Our first example is a very simple modification of the classical Dutch Book Theorem given above. Namely, if we replace $\mathbb{V}^T$ by a non-empty subset, $\mathbb{V}^-$ say, of this set, effectively then saying that only certain worlds (valuations) are possible, then any $B \in \mathbb{B}$ avoiding a Dutch Book must be a convex combination of these valuations, equivalently must be a probability function on SL which assigns zero probability to the atoms α_V for $V \in \mathbb{V}^T - \mathbb{V}^-$ (and conversely).

For a second example, which is an already well known result due to Jaffray, [8], (see also [13]) suppose that our possible worlds might not yet be fully created (maybe it's still only the fifth day, or maybe these are worlds in a book where some details have not been made explicit) so that in any world our total knowledge might consist of knowing the truth of some non-contradictory $\theta \in SL$ (and hence all logical consequences of θ) where θ is not necessarily an atom. So it may be that not all questions of classical truth and falsity are determined. In that case if we were to identify true with 'known true' in the classical sense (with falsity equated with not known to be true) then our set of valuations, denoted $\mathbb{V}^D$ would consist of $\{ V_\theta \mid \theta \in SL \}$ where for $\theta \in SL$ V_θ is the function from SL to $\{0,1\}$ such that

$$V_\theta(\phi) = 1 \iff \theta \models \phi.$$

In this case by results of Shafer the functions $B \in \mathbb{B}$ which are convex combinations of functions in $\mathbb{V}^D$ (equivalently by Theorem 2 avoid a Dutch Book) are precisely the *Dempster-Shafer belief functions* on SL (see [14] or, in the notation of this note, [10]). That is functions $B \in \mathbb{B}$ such that for all $\theta, \phi, \theta_1, ..., \theta_m \in SL$

(DS1) If $\models \theta$ then $B(\theta) = 1$, $B(\neg\theta) = 0$,
(DS2) If $\models (\theta \leftrightarrow \phi)$ then $B(\theta) = B(\phi)$,
(DS3) $B(\bigvee_{i=1}^m \theta_i) \geq \sum_S (-1)^{|S|-1} B(\bigwedge_{i \in S} \theta_i)$,

where in (DS3) S ranges over the non-empty subsets of $\{1, 2, ..., m\}$.

Again we might note that as with probability the 'hard' part of this Dutch Book result is in Shafer's derivation of the equivalence of convex combinations of functions in $\mathbb{V}^D$ with Dempster-Shafer belief functions.

For our next examples we point out a generalization of the conclusions following Theorem 2 to certain logics containing, possibly as derived connectives, conjunction and disjunction satisfying (T2) and (T3). Precisely let $\mathcal{L}$ be a finite propositional language with at least the connectives $\wedge, \vee$ (and possibly also modalities). Let $S\mathcal{L}$ be the set of sentences of $\mathcal{L}$ and suppose that the logic carries with it a notion of a valuation (world, interpretation) $V : S\mathcal{L} \longrightarrow \{0,1\}$. Let $\mathbb{V}^{\mathcal{L}}$ be the set of such valuations of $\mathcal{L}$. As usual for $\theta \in S\mathcal{L}$, $\Gamma \subseteq S\mathcal{L}$ define

$$\Gamma \models_{\mathcal{L}} \theta \iff \forall V \in \mathbb{V}^{\mathcal{L}}, \text{ if } V(\phi) = 1 \text{ for all } \phi \in \Gamma \text{ then } V(\theta) = 1.$$

As usual we write $\models_{\mathcal{L}} \theta$ (corresponding to $\Gamma = \emptyset$) if θ is an $\mathcal{L}$-tautology, i.e. $V(\theta) = 1$ for all $V \in \mathbb{V}^{\mathcal{L}}$, and write $\theta \models_{\mathcal{L}}$ if θ is an $\mathcal{L}$-contradiction, i.e. $V(\theta) = 0$ for all $V \in \mathbb{V}^{\mathcal{L}}$. Let $\mathbb{B}^{\mathcal{L}}$ be the set of functions from $S\mathcal{L}$ into $[0,1]$. The following theorem is already covered by the general theorem 41.1 of Choquet's seminal [3] but again for completeness we give a direct, elementary, proof.

Theorem 5 *Let $B \in \mathbb{B}^{\mathcal{L}}$ and suppose that for $V \in \mathbb{V}^{\mathcal{L}}$, (T2) and (T3) hold. Then $B \restriction \Gamma$ is a convex combination of functions in $\{V \restriction \Gamma \mid V \in \mathbb{V}^{\mathcal{L}}\}$ for every finite $\Gamma \subset SL$ if and only if B satisfies that for all $\theta, \phi \in S\mathcal{L}$,*

- ($\mathcal{L}$1) If $\models_{\mathcal{L}} \theta$ then $B(\theta) = 1$, if $\theta \models_{\mathcal{L}}$ then $B(\theta) = 0$,
- ($\mathcal{L}$2) If $\theta \models_{\mathcal{L}} \phi$ then $B(\theta) \leq B(\phi)$,
- ($\mathcal{L}$3) $B(\theta \vee \phi) + B(\theta \wedge \phi) = B(\theta) + B(\phi)$.

Proof. Clearly ($\mathcal{L}$1-3) hold for $V \in \mathbb{V}^{\mathcal{L}}$ and hence for any convex combination of such functions.

To prove the other direction let Γ be a finite subset of $S\mathcal{L}$ which is closed under subformulae. Now for each $\emptyset \neq \Delta \subset \Gamma$ set

$$a_\Delta = B\left(\bigwedge_{\theta \in \Delta} \theta\right) - B\left(\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi\right)$$

and pick, if possible, a $V_\Delta \in \mathbb{V}^{\mathcal{L}}$ such that

$$V_\Delta\left(\bigwedge_{\theta \in \Delta} \theta\right) = 1, \quad V_\Delta\left(\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi\right) = 0.$$

Notice that by ($\mathcal{L}$2) $a_\Delta \geq 0$ since

$$\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi \models_{\mathcal{L}} \bigwedge_{\theta \in \Delta} \theta.$$

Notice also that if no such V_Δ exits then

$$\bigwedge_{\theta \in \Delta} \theta \models_{\mathcal{L}} \bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi \models_{\mathcal{L}} \bigwedge_{\theta \in \Delta} \theta,$$

so by $(\mathcal{L}2)$ $a_\Delta = 0$.

For $\Delta = \emptyset$ let

$$a_\Delta = 1 - B\left(\bigvee_{\phi\in\Gamma}\phi\right)$$

and pick, if possible, $V_\Delta \in \mathbb{V}^{\mathcal{L}}$ such that $V(\phi) = 0$ for all $\phi \in \Gamma$. Notice that by $(\mathcal{L}1)$ such a V_Δ does exist if $a_\Delta > 0$.

Finally for $\Delta = \Gamma$ define

$$a_\Delta = B\left(\bigwedge_{\theta\in\Gamma}\theta\right)$$

and pick, if possible $V_\Delta \in \mathbb{V}^{\mathcal{L}}$ such that $V(\theta) = 1$ for all $\theta \in \Gamma$. Again by $(\mathcal{L}1)$ such a V_Δ does exist if $a_\Delta > 0$.

Then, by induction on the size, $|\Gamma|$, of Γ,

$$\begin{aligned}\sum_{\Delta\subseteq\Gamma, a_\Delta>0} a_\Delta &= \sum_{\Delta\subseteq\Gamma} a_\Delta = \\ &\sum_{\emptyset\neq\Delta\subset\Gamma}\left\{B\left(\bigwedge_{\theta\in\Delta}\theta\right) - B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Gamma-\Delta}\phi\right)\right\} \\ &+1 - B\left(\bigvee_{\phi\in\Gamma}\phi\right) + B\left(\bigwedge_{\theta\in\Gamma}\theta\right) \\ = \;& 1\end{aligned}$$

In more detail, this result is true for $|\Gamma| = 1$ whilst for $|\Gamma| > 1$, say $\Gamma = \Omega\cup\{\chi\}$ where $\chi \notin \Omega$, the term

$$\sum_{\emptyset\neq\Delta\subset\Gamma}\left\{B\left(\bigwedge_{\theta\in\Delta}\theta\right) - B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Gamma-\Delta}\phi\right)\right\}$$

can, using $(\mathcal{L}2)$, be written as

$$\begin{aligned}&B(\chi) - B\left(\chi\wedge\bigvee_{\phi\in\Omega}\phi\right) + B\left(\bigwedge_{\theta\in\Omega}\theta\right) - B\left(\bigwedge_{\theta\in\Omega}\theta\wedge\chi\right) \\ &+\sum_{\emptyset\neq\Delta\subset\Omega}\left\{B\left(\chi\wedge\bigwedge_{\theta\in\Delta}\theta\right) - B\left(\chi\wedge\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Omega-\Delta}\phi\right)\right\} \\ &+\sum_{\emptyset\neq\Delta\subset\Omega}\left\{B\left(\bigwedge_{\theta\in\Delta}\theta\right) - B\left(\bigwedge_{\theta\in\Delta}\theta\wedge(\chi\vee\bigvee_{\phi\in\Omega-\Delta}\phi)\right)\right\}.\end{aligned}$$

Using $(\mathcal{L}2)$ and $(\mathcal{L}3)$ on this final term gives

$$B\left(\bigwedge_{\theta\in\Delta}\theta\wedge(\chi\vee\bigvee_{\phi\in\Omega-\Delta}\phi)\right)=$$
$$B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\chi\right)+B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Omega-\Delta}\phi\right)$$
$$-B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\chi\wedge\bigvee_{\phi\in\Omega-\Delta}\phi\right)$$

Making this substitution and cancelling terms gives the original expression with Ω in place of Γ and the result follows by induction.

Similarly, by induction on the size of Γ again, we have that for $\psi\in\Gamma$,

$B(\psi)=B\left(\bigwedge_{\theta\in\Gamma}\theta\right)$
$+\sum_{\psi\in\Delta\subset\Gamma}\left\{B\left(\bigwedge_{\theta\in\Delta}\theta\right)-B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Gamma-\Delta}\phi\right)\right\}$
But this righthandside equals

$$\sum_{\psi\in\Delta\subseteq\Gamma}a_\Delta=\sum_{\psi\in\Delta\subseteq\Gamma,\ a_\Delta>0}a_\Delta=\sum_{\Delta\subseteq\Gamma,\ a_\Delta>0}a_\Delta V_\Delta(\psi)$$

and hence, on Γ, B is, as required, the convex combination $\sum_{a_\Delta>0}a_\Delta V_\Delta$ of elements of $\mathbb{V}^{\mathcal{L}}$. ■

Notice that as a consequence of the *proof* of this result, given any finite subset Γ of $S\mathcal{L}$ we can precisely specify a finite $\Gamma'\subseteq S\mathcal{L}$ such that if B satisfies $(\mathcal{L}1)$,$(\mathcal{L}2)$,$(\mathcal{L}3)$ whenever $\theta,\phi\in\Gamma'$ then there is no Dutch Book against B with the $\theta_1,...,\theta_m$ restricted to being elements of Γ. Or putting it another way if $\theta_1,...,\theta_m$ (with suitable stakes) constitute a Dutch Book against B then we can generate a finite set of sentences amongst which is a contradiction to B satisfying $(\mathcal{L}1)$,$(\mathcal{L}2)$,$(\mathcal{L}3)$.

Theorem 5 applies to a number of well known propositional logics, for example the standard modal logics K,T,S_4,S_5 etc. and intuitionistic logic. It also applies to certain paraconsistent logics in which conjunction and disjunction retain their classical interpretation. To expand on this point, one common way to specify a paraconsistent logic (for example in [1]) is to specify a semantics based on a variation of the usual Tarskian interpretation of the connectives. So, for example, in Batens' *Negation Glut* logic $\mathcal{N}$ the set of valuations $\mathbb{V}^{\mathcal{N}}$ are those functions $V:SL\longrightarrow\{0,1\}$ satisfying (T2-4) and, in place of (T1),

(N1) $\quad V(\theta)=0\ \Rightarrow\ V(\neg\theta)=1,$

the resulting consequence relation being defined, as expected, by

$$\Gamma\models_{\mathcal{N}}\theta\iff\forall V\in\mathbb{V}^{\mathcal{N}},\ \text{if } V(\phi)=1 \text{ for all } \phi\in\Gamma \text{ then } V(\theta)=1.$$

In these semantics one might interpret the negation in $\neg\theta$ as asserting that that there were reasons to reject θ, thus allowing the possibility of taking $\neg\theta$ to be true even though one had already assigned θ truth value 1. Corollary 4 now says that according to these semantics (i.e. interpretation of the connectives) it is not possible to construct a Dutch Book against $B \in \mathbb{B}$ just if for every finite subset Γ of SL B restricted to Γ, $B \restriction \Gamma$, is a convex combination of functions in $\{V \restriction \Gamma | V \in \mathbb{V}^{\mathcal{N}}\}$, equivalently by Theorem 5[4] if

($\mathcal{N}1$) If $\models_{\mathcal{N}} \theta$ then $B(\theta) = 1$,
($\mathcal{N}2$) If $\theta \models_{\mathcal{N}} \phi$ then $B(\theta) \leq B(\phi)$,
($\mathcal{N}3$) $B(\theta \vee \phi) + B(\theta \wedge \phi) = B(\theta) + B(\phi)$.

It is worth observing that the conditions (DS1) and (DS2) for a Dempster-Shafer belief function could be replaced, by analogy to $(\mathcal{L}1), (\mathcal{L}2)$, with the conditions

(1) If $\models_{DS} \theta$ then $B(\theta) = 1$,
if $\theta \models_{DS}$ then $B(\theta) = 0$,
(2) If $\theta \models_{DS} \phi$ then $B(\theta) \leq B(\phi)$,

since, with the obvious meaning, $\models_{DS}$ equals classical logical consequence. Note however that the V_θ in $\mathbb{V}^D$ do not necessarily satisfy condition (T3). In turn then Jaffray's result hints that perhaps an analog of Theorem 5 might also be provable without assuming (T3).

For our last example we show that Theorem 2 may also be applicable in the other direction to provide a 'justification in terms of betting' for certain 'belief functions'[5] even where their original formulation made no direct reference to 'possible worlds'.

The idea of an *Ent belief function* was introduced (except for some minor modifications) in [11] and might be loosely described as qualitative *case-based* belief. Unlike belief as probability Ent belief functions can be seen to arise naturally from an agent's past experiences and are not dogged by the problems of computational infeasibility and internal inconsistencies commonly associated with real belief as probability. On the other hand Ent belief values can be viewed as *imprecise probabilities* in the sense that if an agent's past experiences are determined by a fixed probability function then the corresponding Ent belief function tends to this in the limit.

For the purpose of this note we may take Ent belief functions on SL to be characterized (see [11],[12],[2] for precise details) as those functions $B : SL \to [0, 1]$ which satisfy that for $\theta, \phi, \psi, \chi, \chi' \in SL$,

[4] In these semantics there are no $\mathcal{N}$-contradictions.

[5] This, of course, could also be said to be the case for the second example above concerning Dempster-Shafer belief functions.

(E1) If $\models \theta$ then $B(\theta) = 1$.
(E2) $B(\theta \vee \phi) = B(\theta) + B(\neg\theta \wedge \phi)$.
(E3) If $B(\theta \wedge \phi) > 0$ then $B(\phi \wedge \theta) > 0$.
(E4) $B(\chi) = B(\chi')$ whenever χ' is the result of (respectively) replacing a subformula of χ of the form

$$\left.\begin{array}{l} \neg\neg\theta \\ \neg(\theta \wedge \phi) \\ \theta \wedge \neg\theta \\ \theta \wedge (\psi \wedge \theta) \\ \theta \wedge (\phi \wedge \psi) \\ (\theta \vee \phi) \wedge \psi \\ \theta \wedge (\phi \wedge \psi) \\ \theta \vee \psi \\ \theta \wedge \theta \\ (\psi \vee \neg\psi) \vee \theta \end{array}\right\} \text{ by } \left\{\begin{array}{l} \theta \\ \neg\theta \vee \neg\psi \\ \neg\theta \wedge \theta \\ \theta \wedge \psi \\ (\theta \wedge \phi) \vee (\theta \wedge \psi) \\ (\theta \wedge \psi) \vee (\neg\theta \wedge (\phi \wedge \psi)) \\ (\theta \wedge \phi) \wedge \psi \\ \theta \vee (\neg\theta \wedge \psi) \\ \theta \\ \psi \vee \neg\psi \end{array}\right.$$

Given such belief functions it is natural, in the context of this paper, to ask if they can also be justified in terms of avoiding a Dutch Book for *some* notion of 'a world'. As observed by Richard Booth and the author this essentially follows from lemma A.8 of [11]. Namely, worlds can be thought of as the product of lazy and capricious gods who only bother to decide the truth or falsity of a propositional variable when specifically required to do so. [One might compare this to a situation where, say the laws of physics, were only determined at the stage at which the critical experiments were conducted!] Thus to decide the truth or falsity of a sentence θ the god simply reads θ from left to right filling in truth values for the propositional variables as s/he goes along (depending only on the order in which they appear, so not on θ per se). With this notion of a world/valuation the results in [11] show that $B : SL \to [0, 1]$ satisfying (E3) avoids a Dutch Book if and only if B is an Ent belief function in the sense of satisfying (E1)–(E4).

3 Conclusion

In this short note we have pointed out that the 'Dutch Book Method' is applicable not just to classical propositional logic, where it yields probability as the only rational belief (as willingness to bet) but also to a range of non-standard logics with alternate notions of truth and possible worlds. For these it in turn produces a spectrum of 'rational' belief functions.

Whilst we have focused on two-valued semantics Theorem 2 is equally applicable in the case of logics with, for example, truth values in the real interval $[0, 1]$, again the 'tricky' part is finding the 'local' equivalent of being a convex

combination of valuations/worlds. In this regard it is shown in [7] that if in the Łukasiewicz logic Ł$_{k+1}$ (i.e. the possible truth values are $0, 1/k, 2/k, ..., (k-1)/k, 1$ and negation, disjunction, conjunction, implication are defined respectively by the truth tables/functions, $1-x$, $\min\{1, x+y\}$, $\max\{0, x+y-1\}$, $\min\{1, 1-x+y\}$) B avoids a Dutch Book then B satisfies

(Ł$_{k+1}$1) If $\models_{Ł_{k+1}} \theta$ then $B(\theta) = 1$,
if $\theta \models_{Ł_{k+1}}$ then $B(\theta) = 0$,

(Ł$_{k+1}$3) $B(\theta \vee \phi) + B(\theta \wedge \phi) = B(\theta) + B(\phi)$.

As subsequently observed, the converse of this result is also true by the analogue of Theorem 2 since by a result in [6] functions B satisfying (Ł$_{k+1}$1,3) are convex combinations of Ł$_{k+1}$-valuations. Whether or not similar results hold for other 'fuzzy logics' (and in particular for Ł$_\infty$) apparently awaits clarification.

Appendix

Let $\vec{b} =< B(\theta_1), ..., B(\theta_m) >$ and let $\vec{c}$ be the closest point in Y to $\vec{b}$. Since Y is closed convex there is a unique such point and since $\vec{b} \notin Y$, $\vec{b} \neq \vec{c}$. Let $\vec{e} = \frac{1}{2}(\vec{b}+\vec{c})$ and let H be the plane of vectors $\vec{x}$ such that

$$(\vec{x} - \vec{e}) \cdot (\vec{b} - \vec{e}) = 0.$$

Then Y cannot intersect H, since if it did, at $\vec{a}$ say, then there would be a point on the line between $\vec{a}$ and $\vec{c}$ in Y and closer to $\vec{b}$ than $\vec{c}$. Hence Y must lie entirely on the opposite side of H from $\vec{b}$ and

$$\vec{b} \cdot (\vec{e} - \vec{b}) < 0 < \vec{v} \cdot (\vec{e} - \vec{b})$$

for all $\vec{v} \in \mathbb{V}$, giving

$$(\vec{v} - \vec{b}) \cdot (\vec{b} - \vec{e}) < 0,$$

as required.

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