

Scaled Boolean Algebras

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Scaled Boolean algebras are a category of mathematical objects that arose from attempts to understand why the conventional rules of probability should hold when probabilities are construed, not as frequencies or proportions or the like, but rather as degrees of belief in uncertain propositions. This paper separates the study of these objects from that not entirely mathematical problem that motivated them. That motivating problem is explicated in the first section, and the application of scaled Boolean algebras to it is explained in the last section. The intermediate sections deal only with the mathematics. It is hoped that this isolation of the mathematics from the motivating problem makes the mathematics clearer.

Key Words: scaled Boolean algebras, epistemic probability theory, justification of Bayesianism, comparative probability orderings, qualitative probability

1. INTRODUCTION

1.1. First glimpse

Q: Why should the conventional rules of probability hold when probabilities are assigned, not to *events* that are *random* according to their *relative frequencies* of occurrence, but rather to *propositions* that are *uncertain* according to the degree to which they are supported by the evidence?

A: Because probability measures should preserve both the logical partial ordering of propositions (ordered by logical implication) and the operation of relative negation.

The explanation and justification of this proposed answer are not entirely mathematical and appear in §8 — the last section of this paper. Our main concern will be the mathematical theory that the answer motivates: the theory of mappings that, like probability measures and Boolean isomorphisms, preserve partial orderings and certain kinds of relative complementations.

1.2. What are scales and what do they measure?

A scale, as we shall use that word, amounts to a partially ordered set with what will be called an “additive relative complementation.” In some ways, these behave like lattice-theoretic relative complementation, although some of the posets on which they are defined are not lattices, and when they are lattices, the additive relative complementation and the lattice-theoretic relative complementation usually differ. Additive relative complementation shares the following two properties with lattice-theoretic relative complementation: If $\alpha \leq \beta \leq \gamma$, then the additive complement $\sim \beta_{[\alpha, \gamma]}$ of β relative to the interval $[\alpha, \gamma]$ goes down from γ to α , as β goes up from α to γ , and the additive complement of the additive complement of β is β (both complements being taken relative to the same interval $[\alpha, \gamma]$). Indeed, when the scale is a Boolean algebra, then the additive relative complementation and the lattice-theoretic relative complementation are the same. But when the scale is an interval $[\alpha, \gamma]$ on the real line, which is a lattice with no lattice-theoretic relative complementation, then the additive relative complementation is $\beta \mapsto \sim \beta_{[\alpha, \gamma]} = \gamma - \beta + \alpha$.

In lattices (and in particular, in Boolean algebras) two operations with which conventional lattice-theoretic relative complementation neatly meshes are each other’s duals: the meet and the join. In scales, two operations with which additive relative complementation neatly meshes are again duals of each other: We shall call them addition (+) and dual-addition (\oplus). When the scale is a Boolean algebra, then addition and dual addition coincide exactly with meet and join. When the scale is the interval $[0, 1]$ on the real line, then addition is ordinary addition restricted to pairs of numbers whose ordinary sum is within the interval $[0, 1]$, and the dual addition is $(\alpha, \beta) \mapsto \alpha + \beta - 1$, again restricted to pairs of numbers for which the value of that operation is within the interval. Two de-Morganesque laws relate additive relative complementation to addition and dual addition, and a “modular law” says $(\zeta + \eta) \oplus \theta = \zeta + (\eta \oplus \theta)$ when ζ, η, θ are suitably situated. This modular law is reminiscent of the law defining modular lattices, which says $(x \vee y) \wedge z = x \vee (y \wedge z)$ whenever $x \leq z$ (where “ \wedge ” and “ \vee ” are meet and join).

All scales live somewhere between (i.e., *inclusively* between) the Boolean case and various sorts of linearly ordered cases.

Probability, construed somewhat liberally, will be measured on linearly ordered scales — we will allow some generalizations of probability measures to take values in other linearly ordered scales than the interval $[0, 1]$ on the real line. These generalizations will, like probability measures, preserve the partial order and the relative complementation. I think I chose the word “scale” because of talk among followers of Edwin Jaynes about “scales” on which probability can be measured. They were inspired by a theorem from the physicist Richard T. Cox’s book *Algebra of Probable Inference* [1]

that under certain semi-plausible (and too strong, in my view) assumptions about the behavior of probabilities, construed as degrees of belief in uncertain propositions, all such scales must be isomorphic to the unit interval $[0, 1] \subseteq \mathbb{R}$ with its usual furniture – addition, multiplication, and linear ordering – and probability measures must conform to the usual addition and multiplication rules.

This paper resulted from my attempt to understand Cox’s arguments, but what we do here will be quite different from those, to say the least.

1.3. Frequentism and Bayesianism

“Frequentists” assign probabilities to random events according to their relative frequencies of occurrence, or to subsets of populations as proportions of the whole. “Bayesians”, on the other hand, assign probabilities to uncertain propositions according to the degree to which the evidence supports them. Frequentists treat probabilities as *intrinsic* to the object of study, and Bayesians treat them as *epistemic*, i.e., conditional on one’s *knowledge* of the object of study. Frequentist and Bayesian methods of statistical inference differ, and their relative merits have been debated for decades.

Here is a poignant example of a problem whose space of feasible solutions changes when the Bayesian outlook replaces the frequentist one. The respective prices of three kinds of gadgets are \$20, \$21, and \$23. Records of the gross receipts of Acme Gadgets for the year 2099 show that customers bought 3,000,000,000,000,000 gadgets during that year, spending \$66,000,000,000,000,000 on them, so that they spent an average of \$22 per gadget. One of those gadgets sits in an unopened box on your desk. It is just as likely to be any of the 3,000,000,000,000,000 gadgets as it is to be any of the others.

Ambiguous question: Given this information, what are the probabilities that this gadget is of the first, second, and third kinds?

A frequentist way to construe the question: What proportions p_1, p_2, p_3 , of the first, second, and third kinds, respectively, were purchased?

A Bayesian way to construe the question: What degrees of belief p_1, p_2, p_3 should we assign to the propositions that *this particular* gadget is respectively of the first, second, or third kind, if we have only the information given above and no more?

Under the frequentist construction of the question, the feasible solutions are the convex combinations of the two extreme solutions

$$\begin{aligned}(p_1, p_2, p_3) &= (0, 1/2, 1/2), \\ (p_1, p_2, p_3) &= (1/3, 0, 2/3).\end{aligned}$$

But both of these extreme solutions are incorrect under the Bayesian construction of the question! The first extreme solution says $p_1 = 0$. By the Bayesian construction, this would mean that, on the meager evidence given, we can be sure that the gadget in the box is not of the first kind. But obviously we cannot. The other extreme solution says $p_2 = 0$, and is defeated by the same consideration. Whether there is any solution, and whether there is only one solution, on the Bayesian construction, is a subtler question. (Edwin Jaynes' "principle of maximum entropy," proposed in [8], entails that the rationally justified degrees of belief are those that maximize the entropy $\sum_{i=1}^3 -p_i \log p_i$ subject to the constraints $p_1, p_2, p_3 \geq 0$, $p_1 + p_2 + p_3 = 1$, and $\$20p_1 + \$21p_2 + \$23p_3 = \22 .)

1.4. Why the conventional rules of probability?

The conventional mathematical rules of probability include additivity and definitions and other characterizations of conditional probability. To "pure" mathematicians, these are merely axioms or definitions. To frequentists, finite additivity and the definition of conditioning on events of positive probability are trivially true propositions about frequencies or proportions. But to Bayesians, the rules of probability are problematic.

In [1] Cox addressed the question of why finite additivity – the "sum rule" – and the conventional definition of conditional probability – the "product rule" – should be adhered to if probabilities are taken to be degrees of belief rather than proportions or frequencies or the like.

1.5. How and why this work came about

I set out to recast Cox's argument in a more abstract form. However, the project went in a direction of its own choosing. Ultimately I had a different, but not more abstract, argument (call it the "concrete version") for a similar but more extensive set of conclusions, relying on a different (and more plausible, in my view) set of assumptions about rational degrees of belief. Then I set out to recast *that* argument in more abstract language (the "more abstract version") in order to separate the part of it that is purely mathematical from the rest. To that end, I conceived the idea of a scale.

The concrete version of the argument rests on an explanation of why probabilities, construed as degrees of belief in uncertain propositions, should be assigned in a way that preserves the logical partial ordering and relative complementation of propositions. That explanation is not entirely mathematical, and appears in §8.

1.6. Relation of this to earlier work

The concrete version overlaps with some earlier work of Leonard Jimmie Savage [11], Terrence Fine [3], Peter C. Fishburn [4], Bruno de Finetti [2],

Bernard O. Koopman [9], and Kraft, Pratt, and Seidenberg [10] that I had initially largely ignored because those authors seemed to be assuming a weak sort of additivity as an axiom rather than trying to prove that probabilities should be assigned additively. That weak additivity statement appears here as Lemma 4.1. I found that I had rediscovered the result that appears here as Theorem 4.1.

However, a seemingly trivial change in emphasis makes it possible to go considerably beyond where those authors left off, and hence to define the concept of a scale and its addition, dual addition, and additive relative complementation. Those authors considered two orderings of propositions: the first “ \leq ” is the usual logical partial ordering, so that $x \leq y$ if x logically entails y . The second, a linear ordering, “ \preceq ”, is a comparative probability ordering, so that “ $x \preceq y$ ” means x is no more probable than y . Those authors assumed:

$$\begin{aligned} &\text{If } x \leq y \text{ then } x \preceq y, \text{ so “} \preceq \text{” is a linear extension of “} \leq \text{”}. \\ &\text{If } x \preceq y \text{ and } y \wedge z = 0, \text{ then } x \vee z \preceq y \vee z. \quad (\text{weak additivity}) \end{aligned} \tag{1}$$

The seemingly trivial change in emphasis is from an ordering of propositions by probabilities to a suitably well-behaved mapping, called a scaling, from a Boolean algebra of propositions into a partially ordered space of generalized “probabilities.”

Kraft, Pratt, and Seidenberg [10] found that a condition called “strong additivity” is sufficient to guarantee that a comparative probability ordering \preceq has an “agreeing measure,” i.e., a probability measure (in the usual sense of the term) μ , such that $x \preceq y$ holds if and only if $\mu(x) \leq \mu(y)$. We give an enormously simpler sufficient condition called “divisibility” in §5. In [12], Dana Scott proved Kraft, Pratt and Seidenberg’s result by a more generally applicable method. Scott showed that his method can be applied not only to probabilities, but also to other things naturally measured on partially ordered scales.

Unlike the authors cited above, we also consider infinite Boolean algebras. When the Boolean algebra that is the domain of a scaling is infinite, it makes sense to speak of continuity or discontinuity of a scaling at a particular Boolean homomorphism. We shall see that continuity at all homomorphisms whose kernels are principal ideals is the same as complete additivity, and continuity at all 2-valued homomorphisms entails a kind of Archimedeanism.

2. BOOLEAN ALGEBRAS AND SCALES

2.1. Boolean algebras

DEFINITION 2.1. A **Boolean algebra** consists of an underlying set \mathbb{A} with two distinguished elements $0 \neq 1$, two binary operations $(x, y) \mapsto x \wedge y =$ the “meet” of x and y , and $(x, y) \mapsto x \vee y =$ the “join” of x and y , and a unary operation $x \mapsto \sim x =$ the “complement” of x , satisfying the following algebraic laws (which are the same laws that are obeyed by the logical connectives “and”, “or”, “not” or the operations of intersection, union, and complementation of sets): For $x, y, z \in \mathbb{A}$ we have

$$\begin{array}{ll}
 x \wedge y = y \wedge x & x \vee y = y \vee x \\
 (x \wedge y) \wedge z = x \wedge (y \wedge z) & (x \vee y) \vee z = x \vee (y \vee z) \\
 x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\
 x \wedge x = x & x \vee x = x \\
 \sim \sim x = x & \\
 \sim(x \wedge y) = (\sim x) \vee (\sim y) & \sim(x \vee y) = (\sim x) \wedge (\sim y) \\
 x \wedge \sim x = 0 & x \vee \sim x = 1 \\
 x \wedge 1 = x & x \vee 0 = x \\
 x \wedge 0 = 0 & x \vee 1 = 1 \\
 \sim 1 = 0 & \sim 0 = 1
 \end{array} \tag{2}$$

By a “convenient abuse of language” we shall use the same symbol \mathbb{A} to refer either to a Boolean algebra or to its underlying set.

Every Boolean algebra has a natural partial ordering “ \leq ” defined by $x \leq y$ iff $x \wedge y = x$, or equivalently $x \vee y = y$. With this ordering, $x \wedge y$ and $x \vee y$ are respectively the infimum and the supremum of the set $\{x, y\}$. The largest and smallest elements of \mathbb{A} are respectively 1 and 0.

DEFINITION 2.2. For $a, b \in \mathbb{A}$, if $a \leq b$ then the **complement** of any $x \in [a, b] = \{u : a \leq u \leq b\}$ **relative** to the interval $[a, b]$ is

$$\sim x_{[a,b]} = a \vee (b \wedge \sim x) = b \wedge (a \vee \sim x).$$

PROPOSITION 2.1. For $a, b \in \mathbb{A}$, if $a \leq b$ then the interval $[a, b] = \{x : a \leq x \leq b\}$ is a Boolean algebra whose meet and join operations are the restrictions to $[a, b]$ of the meet and join operations of \mathbb{A} , and whose complementation is relative to this interval.

The proof is a quick exercise.

When $[a, b] = [0, 1] = \mathbb{A}$ then relative complementation coincides with ordinary complementation. An interval $[a, b] \subsetneq \mathbb{A}$ with this structure is not a “subalgebra” of \mathbb{A} because its complementation differs from that of

\mathbb{A} . An interval $[a, b]$ with this structure will be called a “relative Boolean algebra.”

If $[a, b] \subseteq [c, d] \subseteq [0, 1] = \mathbb{A}$ then the intervals $[c, d]$ and $[0, 1]$ are both larger Boolean algebras of which $[a, b]$ is a subinterval. Should one take $\sim x_{[a,b]}$ to be $a \vee (b \wedge \sim x)$ or $a \vee (b \wedge \sim x_{[c,d]})$? A straightforward computation shows that either yields the same result.

The algebraic laws defining a Boolean algebra are those obeyed by the logical connectives “and”, “or”, “not” that connect propositions. The “0” and “1” in a Boolean algebra correspond respectively to propositions known to be false, and propositions known to be true. The relation $x \leq y$ corresponds to the statement that it is known that if x is true then so is y , although the truth values of x and y may be uncertain. If $0 \leq x < y \leq 1$ then y is closer to being known to be true than x is; x is closer to being known to be false than y is. We shall argue that the definition of “scaling” that will follow, captures laws that should be obeyed by any rational assignment of degrees of belief to propositions.

2.2. Basic scalings and scales

A Boolean algebra has a partial ordering and a relative complementation. A “scale” is a more general kind of object with a partial ordering and an “additive” relative complementation (although, as we shall see, the latter fails to be everywhere-defined in some cases). One of the simplest examples of a scale that is not a Boolean algebra is the interval $[0, 1] \subseteq \mathbb{R}$, in which the additive complement of $\beta \in [\alpha, \gamma]$ relative to the interval $[\alpha, \gamma]$ is $\sim \beta_{[\alpha,\gamma]} = \gamma - \beta + \alpha \in [\alpha, \gamma]$. A “scaling” is a mapping from one scale to another that preserves the partial ordering and the additive relative complementation. A “basic” scaling is one whose domain is a Boolean algebra. Among the simplest basic scalings are Boolean isomorphisms and finitely additive measures.

DEFINITION 2.3.

1. A **basic scaling** is a mapping ρ from a Boolean algebra \mathbb{A} into any partially ordered set, that (a) is strictly increasing, and (b) preserves relative complementations. These two conditions mean:

(i) For $x, y \in \mathbb{A}$, if $x < y$ then $\rho(x) < \rho(y)$, and

(ii) For $x, y \in [a, b] \subseteq \mathbb{A}$, if $\rho(x) < \rho(y)$ then $\rho(\sim x_{[a,b]}) > \rho(\sim y_{[a,b]})$, and if $\rho(x) = \rho(y)$ then $\rho(\sim x_{[a,b]}) = \rho(\sim y_{[a,b]})$. (In particular, if $a = 0$, $b = 1$, and $\rho(x) < \rho(y)$, then $\rho(\sim x) > \rho(\sim y)$ and similarly if “=” replaces “<”.)

(A “scaling” is either a basic scaling as defined here, or a more general kind of scaling to be defined in §2.3.)

2. A **scaled Boolean algebra** is a Boolean algebra endowed with a basic scaling. If ρ is a basic scaling on a Boolean algebra $(\mathbb{A}, 0, 1, \wedge, \vee, \sim)$ then $(\mathbb{A}, 0, 1, \wedge, \vee, \sim, \rho)$ is a scaled Boolean algebra.

3. A **scale** is the image $\mathcal{R} = \{\rho(x) : x \in \mathbb{A}\}$ of a basic scaling ρ , with its partial ordering and an additive relative complementation that it inherits from the scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$. The precise definition of this inherited additive relative complementation involves some perhaps unexpected complications, and appears in §2.3. Lower-case Greek letters will usually be used for members of \mathcal{R} , except that the minimum and maximum members of \mathcal{R} will be called 0 and 1 respectively.

Clearly the restriction of a basic scaling to a relative Boolean algebra $[a, b]$ is also a basic scaling.

2.3. The difficulty with relative complementation

By Definition 2.3 (1ii), a scaling induces a sort of complementation on a scale – one may unambiguously define $\sim\rho(x)$ to be $\rho(\sim x)$. This operation is strictly decreasing and is its own inverse: For α, β in a scale \mathcal{R} , we have $\sim\alpha > \sim\beta$ if $\alpha < \beta$, and $\sim\sim\alpha = \alpha$. It may be tempting to think it is just as easy to define an induced complementation *relative* to an interval. Here is the difficulty. Suppose $x, y \in [a, b] \subseteq \mathbb{A}$. Although the definition of “basic scaling” says that if $\rho(x) = \rho(y)$ then $\rho(\sim x_{[a,b]}) = \rho(\sim y_{[a,b]})$, the extension to relative complements requires something stronger. We need to know the following fact.

Suppose ρ is a basic scaling. If $x \in [a, b]$ and $y \in [c, d]$ and

$$\rho(a) = \rho(c) \leq \rho(x) = \rho(y) \leq \rho(b) = \rho(d)$$

then $\rho(\sim x_{[a,b]}) = \rho(\sim y_{[c,d]})$.

More economically stated: $\rho(\sim x_{[a,b]})$ depends on x , a , and b only through $\rho(x)$, $\rho(a)$, and $\rho(b)$. The proof appears in §4.2.4. This proposition makes the following definition unambiguous.

DEFINITION 2.4. Additive relative complementation on \mathcal{R} is given by

$$\sim\rho(x)_{[\rho(a), \rho(b)]} = \rho(\sim x_{[a,b]}).$$

The word “additive” is used because of the relationship between this relative complementation and the operations of addition and dual addition. Those operations are introduced in §4. Additive relative complementation may fail to be everywhere-defined, since it can happen that $\zeta < \eta < \theta$ even while no $x < y < z$ exist in \mathbb{A} whose respective images under ρ are ζ, η, θ . Concrete instances will appear in §3. Scalings for which this happens are

not “divided.” That concept is defined in §5, which section also prescribes the remedy to this pathology.

2.4. General definition of scaling

Here is the definition of “scaling” that is more general than that of “basic scaling.”

DEFINITION 2.5. A **scaling** is a mapping ρ from a scale \mathcal{R} into a partially ordered set, that (a) is strictly increasing, and (b) preserves additive relative complementations. These two conditions mean:

1. For $\zeta, \eta \in \mathcal{R}$, if $\zeta < \eta$ then $\rho(\zeta) < \rho(\eta)$, and
2. For $\zeta, \eta \in [\alpha, \beta] \subseteq \mathcal{R}$, if $\rho(\zeta) < \rho(\eta)$ then $\rho(\sim \zeta_{[\alpha, \beta]}) > \rho(\sim \eta_{[\alpha, \beta]})$, and if $\rho(\zeta) = \rho(\eta)$ then $\rho(\sim \zeta_{[\alpha, \beta]}) = \rho(\sim \eta_{[\alpha, \beta]})$.

Although in this definition ρ need not be a basic scaling, its image is nonetheless the image of a basic scaling $\rho \circ \sigma$, where \mathcal{R} is the image of a basic scaling σ on some Boolean algebra \mathbb{A} . Therefore all images of scalings are images of basic scalings, and we need not change the definition of “scale” as the image of a basic scaling.

DEFINITION 2.6. If $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is a basic scaling and $\sigma : \mathcal{R} \rightarrow \mathcal{S}$ is a scaling, then the basic scaling $\sigma \circ \rho : \mathbb{A} \rightarrow \mathcal{R}$ is an **extension** of the basic scaling ρ .

Why an “extension”? Because σ may map two incomparable members α, β of \mathcal{R} to members $\sigma(\alpha), \sigma(\beta)$ that are comparable either because they are equal (so that σ is not one-to-one) or because a strict inequality holds between them. In other words, σ extends the partial ordering by making comparable, and possibly even equal, things that were incomparable before the extension. (Note that the definition precludes unequal but comparable elements of \mathcal{R} having the same image under σ .)

2.5. Measures on Boolean algebras

DEFINITION 2.7. A **finitely additive measure** on a Boolean algebra \mathbb{A} is a mapping $\rho : \mathbb{A} \rightarrow [0, \infty) \subseteq \mathbb{R}$, satisfying

1. For all $x \in \mathbb{A}$, if $x > 0$ then $\rho(x) > 0$; and
2. For all $x, y \in \mathbb{A}$, if $x \wedge y = 0$, then $\rho(x \vee y) = \rho(x) + \rho(y)$.

The requirement that if $x > 0$ then $\rho(x) > 0$ excludes analogs of non-empty sets of measure zero. We deal with things like Lebesgue measure by regarding sets that differ only by a set of measure zero as equivalent to each other, and considering the Boolean algebra of equivalence classes.

Definition 4.3 will generalize Definition 2.7 by defining the concept of a finitely additive measure whose domain is an arbitrary scale.

3. EXAMPLES

EXAMPLE 3.1. Every isomorphism from one Boolean algebra into another is a basic scaling; hence every Boolean algebra is a scale.

EXAMPLE 3.2. Every finitely additive measure on a Boolean algebra is a basic scaling. Since for any $a \in \mathbb{R}$, $a > 0$, there exist measures whose image is the whole interval $[0, a] \subseteq \mathbb{R}$, that interval is a scale with relative complementation given by $\sim \beta_{[\alpha, \gamma]} = \gamma - \beta + \alpha$ for $\beta \in [\alpha, \gamma]$.

EXAMPLE 3.3. Let \mathcal{M} be a nonempty convex set of finitely additive measures on a Boolean algebra \mathbb{A} . Call $x, y \in \mathbb{A}$ equivalent if \mathcal{M} does not separate x from y , i.e., $\mu(x) = \mu(y)$ for every $\mu \in \mathcal{M}$. Let $\rho(x)$ be the equivalence class to which x belongs. Say that $\rho(x) < \rho(y)$ if for every $\mu \in \mathcal{M}$ we have $\mu(x) \leq \mu(y)$ and for some $\mu \in \mathcal{M}$ the inequality is strict. Plainly this is an antisymmetric relation; to show that it is a strict partial ordering we need to show that it is transitive. If $\rho(x) < \rho(y)$ and $\rho(y) < \rho(z)$ then there exist $\mu, \nu \in \mathcal{M}$ such that $\mu(x) < \mu(y) \leq \mu(z)$ and $\nu(x) \leq \nu(y) < \nu(z)$. Convexity implies $\pi = (\mu + \nu)/2 \in \mathcal{M}$, and then we have $\pi(x) < \pi(z)$. The reader can check that ρ is a scaling. If $\mathcal{M} \neq \emptyset$ is the set of *all* measures on \mathbb{A} then ρ is just the canonical automorphism of \mathbb{A} . At the opposite extreme, \mathcal{M} could contain just one measure and ρ would be a linearly ordered scale.

EXAMPLE 3.4. This example is (1) a simple “non-Archimedean” scale; (2) a scale that is not a lattice; (3) a scaling that is not countably additive, and (4) a scaling that is discontinuous at some 2-valued Boolean homomorphisms on its domain. Precise definitions of terms needed to understand these claims will appear in the sequel.

For $A, B \subseteq \mathbb{N} = \{1, 2, 3, \dots\}$, let $|A| \in \{0, 1, 2, 3, \dots, \aleph_0\}$ be the cardinality of A and let $A \setminus B = \{x : x \in A \ \& \ x \notin B\}$. Call two sets A, B equivalent if

$$|A \setminus B| = |B \setminus A| < \aleph_0,$$

i.e., A can be changed into B by deleting finitely many members and replacing them by exactly the same number of others. Let $\rho(A)$ be the equivalence class to which A belongs. Say that $\rho(A) < \rho(B)$ if $|A \setminus B| < |B \setminus A|$. Note that $|A \setminus B| = |B \setminus A|$ does not imply $\rho(A) = \rho(B)$ unless the common value of these two cardinalities is finite. If $|A \setminus B| = |B \setminus A| = \aleph_0$ then $\rho(A)$ and $\rho(B)$ are not comparable.

What does the poset $\mathcal{R} = \{\rho(A) : A \subseteq \mathbb{N}\}$ look like? Every member α of \mathcal{R} except $\rho(\emptyset)$ has an immediate predecessor, a largest member of \mathcal{R} that is $< \alpha$. Call it $\alpha - 1$. Similarly, every member of \mathcal{R} except $\rho(\mathbb{N})$ has an immediate successor $\alpha + 1$, the smallest member of \mathcal{R} that is $> \alpha$. The

range \mathcal{R} is partitioned into “galaxies”

$$\{ \dots, \alpha - 2, \alpha - 1, \alpha, \alpha + 1, \alpha + 2, \dots \},$$

plus an “initial galaxy”

$$\{ \rho(\emptyset), \rho(\emptyset) + 1, \rho(\emptyset) + 2, \dots \}$$

of “finite elements” and a “final galaxy”

$$\{ \dots, \rho(\mathbb{N}) - 2, \rho(\mathbb{N}) - 1, \rho(\mathbb{N}) \}$$

of “cofinite elements.” For any two galaxies that are comparable, in the sense that any member of one is comparable to any member of the other, uncountably many other galaxies are between them, and infinite antichains of galaxies are between them. (An antichain in a partially ordered set is a set of pairwise incomparable elements.)

This mapping ρ is a scaling. For any finite element $\alpha > 0$, one can write \mathbb{N} as a union of subsets whose images under ρ are $\leq \alpha$. But \mathbb{N} cannot be written as a union of finitely many such sets, so we say that \mathcal{R} is a “non-Archimedean” scale.

Via this example it is easy to see why a scale has more structure than its partial ordering. Single out any typical galaxy, and define a mapping by $\alpha \mapsto \alpha + 1$ if α is in that galaxy, and $\alpha \mapsto \alpha$ otherwise. This is clearly a poset-automorphism, but it is not a scale-automorphism since it fails to preserve relative complementation.

This scale is not a lattice, i.e., a set of two members of \mathcal{R} need not have an infimum or a supremum. If $\alpha \leq$ each of two incomparable members of \mathcal{R} , then so is $\alpha + 1$, and similarly for “ \geq ” and $\alpha - 1$.

In this scale the relative complementation is everywhere-defined.

Remark 3.1. Members of this scale can serve as dimensions of subspaces of infinite-dimensional separable Hilbert space.

EXAMPLE 3.5. (A linearly ordered non-Archimedean scale)

First we tersely describe this example in language comprehensible to those who know nonstandard analysis. Then for others we include a two-page crash course in that subject.

Let n be an infinite integer. The $*$ -finitely additive measure on the set of all *internal* subsets of $\{1, \dots, n\}$ that assigns 1 to every one-element set is a scaling. As in the last example, there is one member, 1, of the range of this scale such that $\{1, \dots, n\}$ can be written as the union of subsets each of which is mapped to something ≤ 1 , but $\{1, \dots, n\}$ cannot be written as the union of finitely many such sets. As in the previous example, the scale is partitioned into uncountably many “galaxies,”

$\{\dots, k-1, k, k+1, \dots\}$, plus an initial galaxy $\{0, 1, 2, \dots\}$ and a final galaxy $\{\dots, n-2, n-1, n\}$. But in this case any two galaxies are comparable.

Now for the two-page crash course. (This can be skipped by anyone who wants to take the previous paragraph on faith.) The ordered field \mathbb{R}^* of nonstandard real numbers properly includes the real field \mathbb{R} . Like all ordered fields that properly include \mathbb{R} , this field is “non-Archimedean.” This term, when applied to ordered fields, has a simpler definition than that used by people who study fields of p -adic numbers. It means that some members $x \neq 0$ of \mathbb{R}^* are infinitesimal, i.e.,

$$\underbrace{|x| + \dots + |x|}_{n \text{ terms}} < 1 \text{ for every finite cardinal number } n.$$

The only infinitesimal in \mathbb{R} is 0. Some other members of \mathbb{R}^* – the reciprocals y of the nonzero infinitesimals – are infinite, i.e.,

$$\underbrace{1 + \dots + 1}_{n \text{ terms}} < |y| \text{ for every finite cardinal number } n.$$

The underlying set of the field \mathbb{R}^* is the image of \mathbb{R} under a mapping $A \mapsto A^*$ from subsets A of \mathbb{R} to subsets of \mathbb{R}^* . In every case $A \subseteq A^*$, with equality if and only if A is finite. Sets of the form A^* for some $A \subseteq \mathbb{R}$ are called “standard” subsets of \mathbb{R}^* . The standard sets belong to a much larger class of subsets of \mathbb{R}^* called “internal” sets. Similarly each function $f : A \rightarrow \mathbb{R}$ extends to a function $f^* : A^* \rightarrow \mathbb{R}^*$; these are called “standard” functions, and belong to the much larger class of “internal” functions. Sets and functions that are not internal are “external.” Although space limitations forbid defining these precisely here, their role and importance will become evident from the following proposition and its accompanying examples.

PROPOSITION 3.1. (*The “transfer principle”*)

1. *Suppose a proposition that is true of \mathbb{R} can be expressed via functions of finitely many variables (e.g. $(x, y) \mapsto x + y$), relations among finitely many variables (e.g. $x \leq y$), finitary logical connectives such as “and”, “or”, “not”, “if ... then ...”, and the quantifiers $\forall x \in \mathbb{R}$ and $\exists x \in \mathbb{R}$. (For example, one such proposition is $\forall x \in \mathbb{R} \exists y \in \mathbb{R} x + y = 0$.) Such a proposition is true in \mathbb{R} if and only if it is true in \mathbb{R}^* when the quantifier $\forall x \in \mathbb{R}^*$ replaces $\forall x \in \mathbb{R}$, and similarly for “ \exists ”.*

2. *Suppose a proposition otherwise expressible as simply as those considered in part (1.) above mentions some particular sets $A \subseteq \mathbb{R}$. Such a proposition is true in \mathbb{R} if and only if it is true in \mathbb{R}^* with each such*

“ A ” replaced by the corresponding “ A^* ”. (Here are two examples: (1) The set $[0, 1]^* = \{x \in \mathbb{R} : 0 \leq x \leq 1\}^*$ must be $\{x \in \mathbb{R}^* : 0 \leq x \leq 1\}$, including not only members of \mathbb{R} between 0 and 1 inclusive, but also members of \mathbb{R}^* that differ from those by infinitesimals. To see this, observe that the sentence

$$\forall x \in \mathbb{R} (x \in [0, 1] \text{ if and only if } 0 \leq x \leq 1)$$

is true in \mathbb{R} , and apply the transfer principle. (2) The set \mathbb{N}^* must be a set that has no upper bound in \mathbb{R}^* (since the sentence expressing the non-existence of an upper bound of \mathbb{N} in \mathbb{R} is simple enough for the transfer principle to apply to it) and must contain $n + 1$ if it contains n , but must not contain anything between n and $n + 1$. Members of $\mathbb{N}^* \setminus \mathbb{N}$ are “infinite integers.”)

3. Suppose a proposition otherwise expressible as simply as those considered in parts (1.) and (2.) above contains the quantifier “ $\forall A \subseteq \mathbb{R} \dots$ ” or “ $\exists A \subseteq \mathbb{R} \dots$ ”. Such a proposition is true in \mathbb{R} if and only if it is true in \mathbb{R}^* after the changes specified above and the replacement of the quantifiers with “[\forall internal $A \subseteq \mathbb{R}^* \dots$]” and “[\exists internal $A \subseteq \mathbb{R}^* \dots$]”. (Here are three examples: (1) Every nonempty internal subset of \mathbb{R}^* that has an upper bound in \mathbb{R}^* has a least upper bound in \mathbb{R}^* . Consequently the set of all infinitesimals is external. (2) The well-ordering principle implies every nonempty internal subset of \mathbb{N}^* has a smallest member. Consequently the set $\mathbb{N}^* \setminus \mathbb{N}$ of all infinite integers is external. (3) If n is an infinite integer, then the set $\{1, \dots, n\}$ (which is not standard!) must be internal. To prove this, first observe that the following is trivially true:

$$\forall n \in \mathbb{N} \exists A \subseteq \mathbb{N} \forall x \in \mathbb{N} [x \in A \text{ iff } x \leq n].$$

Consequently

$$\forall n \in \mathbb{N}^* \exists \text{ internal } A \subseteq \mathbb{N}^* \forall x \in \mathbb{N}^* [x \in A \text{ iff } x \leq n].)$$

4. As with internal sets, so with internal functions: Replace “[$\forall f : A \rightarrow \mathbb{R} \dots$]” with “[\forall internal $f : A^* \rightarrow \mathbb{R}^* \dots$]”, and similarly with “ \exists ” in place of “ \forall ”. (For example: If n is an infinite integer, then the complement of the image of any internal one-to-one function f from the infinite set $\{1, \dots, n\}$ into $\{1, \dots, n\} \cup \{n + 1, n + 2, n + 3\}$ has exactly three members. Because of the infiniteness of the domain, the complements of the images of one-to-one functions from the former set to the latter come in many sizes, but most of these functions are external.)

The last example described in Proposition 3.1 motivates a crucial definition:

DEFINITION 3.1. A ***-finite** (pronounced “star-finite”) subset of \mathbb{R}^* is one that can be placed in *internal* one-to-one correspondence with $\{1, \dots, n\}$ for some $n \in \mathbb{N}^*$.

Armed with this definition, readers not previously familiar with nonstandard analysis can go back and read the description of the example.

EXAMPLE 3.6. This scale is not “divided,” but is “divisible” – those terms will be defined in §5. Let \mathbb{A} be the Boolean algebra of all subsets of a set $\{a, b, c\}$, i.e., meet, join, complement are the union, intersection, and complement operations on sets. The set $\mathcal{R} = \{0, \alpha, \beta, \gamma, \delta, 1\}$, partially ordered as in Figure 1, is a scale.

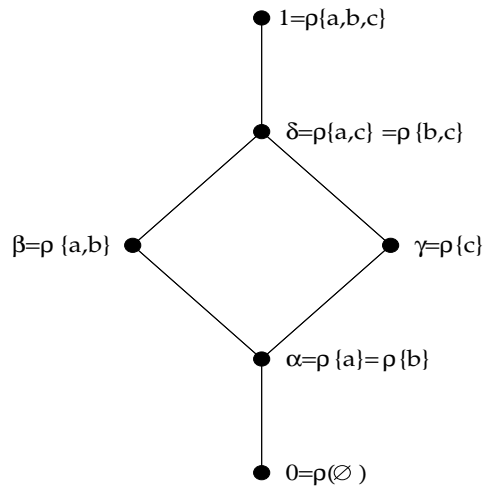


FIG. 1.

EXAMPLE 3.7. Extend the partial ordering of the previous example so that $\beta < \gamma$, making \mathcal{R} linearly ordered. The same mapping into the same set, but with a different ordering of that set, is a different scaling on the same Boolean algebra. This scaling is isomorphic as a scaling to any measure μ on the set of all subsets of $\{a, b, c\}$ satisfying

$$\mu\{a\} = \mu\{b\} \quad \text{and} \quad \mu\{a, b\} < \mu\{c\}.$$

So measures that are not scalar multiples of each other can be isomorphic to each other as scalings.

EXAMPLE 3.8. By now Figure 2 should be self-explanatory. In §4.2.4 this scale will provide us with an example of something that “ought to be” a relative complement but is not.

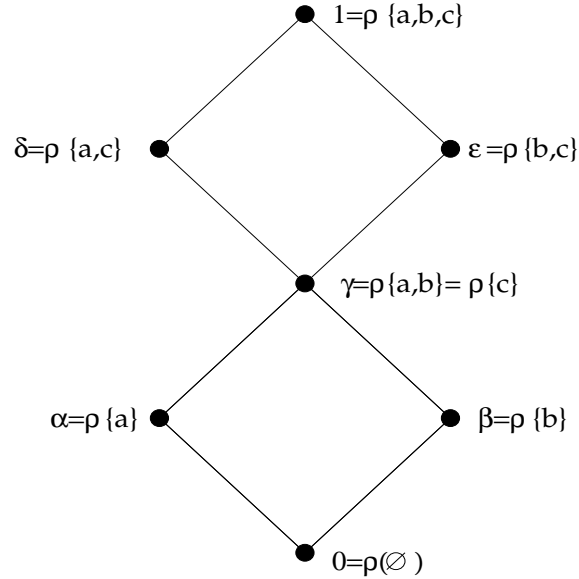


FIG. 2.

EXAMPLE 3.9. Let \mathbb{A} be the Boolean algebra of all subsets of $\{a, b, c\}$, and let \mathcal{R} be the set of all such subsets, partially ordered by saying that $A < B$ iff $|A| < |B|$ (but note that $|A| = |B|$ does not imply $\rho(A) = \rho(B)$). See Figure 3. Let ρ map each subset of $\{a, b, c\}$ to itself. Then ρ is a scaling from \mathbb{A} into \mathcal{R} . One point of this example is that a scaling ρ for which $\rho(a)$, $\rho(b)$, and $\rho(c)$, are pairwise incomparable can nonetheless have a properly more extensive partial ordering than does the domain of the scaling. Another point of this example is that this is another case in which a scale’s relative complementation is obviously not determined by its partial ordering. Finally, this is a finite scale that is not a lattice; for example, there is no smallest element that is \geq both $\{a\}$ and $\{b\}$.

On the Boolean algebra of subsets of $\{a, b, c\}$, up to isomorphism, there exist 17 one-to-one scalings, including two whose images are linearly ordered scales, and 10 scalings that are not one-to-one, including six whose images are linearly ordered scales. The last three examples above are of course among these 27 scalings. On the Boolean algebra of subsets of

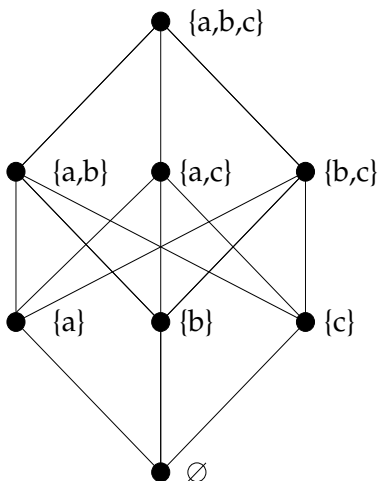


FIG. 3.

$\{a, b, c, d\}$ there are 14 one-to-one scalings whose images are linearly ordered, and many scalings that are less well-behaved.

4. ADDITIVITY AND ITS CONSEQUENCES

4.1. Addition

The following lemma is an easy consequence of Definition 2.3, but to get from this lemma to Theorem 4.1, the result that explains the title of this section, is less straightforward.

LEMMA 4.1. *If $\rho(x) \leq \rho(y)$ and $y \wedge z = 0$, then $\rho(x \vee z) \leq \rho(y \vee z)$, and similarly if “ $<$ ” replaces “ \leq ” throughout.*

Proof. Let $x_1 = x \wedge \sim z \leq x$. Then $x_1, y \in [0, \sim z]$ and $\rho(x_1) \leq \rho(y)$. Consequently $\rho(\sim x_1 [0, \sim z]) \geq \rho(\sim y [0, \sim z])$. This reduces to $\rho(\sim z \wedge \sim x_1) \geq \rho(\sim z \wedge \sim y)$, whence we get $\rho(x_1 \vee z) \leq \rho(y \vee z)$. Since $z \vee x_1 = z \vee x$ we get $\rho(x \vee z) \leq \rho(y \vee z)$. For strict inequalities the proof is similar. ■

The proof of the next result uses Lemma 4.1 three times, but the three parts of the proof are not really parallel to each other.

THEOREM 4.1 (Basic scalings are finitely additive). *If $x \wedge y = 0$ then $\rho(x \vee y)$ depends on x and y only through $\rho(x)$ and $\rho(y)$, and in a strictly increasing fashion. In other words, if $u \wedge v = 0 = x \wedge y$, $\rho(u) = \rho(x)$, and $\rho(v) = \rho(y)$, then $\rho(u \vee v) = \rho(x \vee y)$, and if “ $<$ ” replaces “ $=$ ” in either or*

both of the assumed equalities between values of ρ , then “ $<$ ” replaces “ $=$ ” in the conclusion.

Proof. To prove “ $=$ ” it suffices to prove both “ \leq ” and “ \geq ”. By symmetry we need only do the first. Although the proof that $\rho(u \vee v) \leq \rho(x \vee y)$ must rely on the fact that $\rho(u) \leq \rho(x)$ and $\rho(v) \leq \rho(y)$, and that $x \wedge y = 0$, the assumption that $u \wedge v = 0$ is needed only for proving the inverse inequality “ \geq ”.

Let $u_1 = u \wedge \sim y$, $y_1 = y \wedge \sim u$, $w = u \wedge y$. We use Lemma 4.1 three times – once with u_1 in the role of z , once with y_1 in that role, and once with w in that role.

By definition of u_1 , we have $u_1 \wedge y = 0$. Therefore by Lemma 4.1 we can add u_1 to both sides of the inequality	By hypothesis $x \wedge y = 0$. By definition of y_1 , this implies $x \wedge y_1 = 0$. Therefore by Lemma 4.1 we can add y_1 to both sides of the inequality
$\rho(v) \leq \rho(y)$	$\rho(u) \leq \rho(x)$
to get $\rho(u_1 \vee v) \leq \rho(u_1 \vee y)$.	to get $\rho(u \vee y_1) \leq \rho(x \vee y_1)$.

Since $u_1 \vee y = u \vee y = u \vee y_1$, we get

$$\rho(u_1 \vee v) \leq \rho(x \vee y_1). \tag{3}$$

Before applying Lemma 4.1 the third time, we must check that $(x \vee y_1) \wedge w = 0$. For this we need both the assumption that $x \wedge y = 0$ and the definitions of y_1 and w . Then Lemma 4.1 applied to (3) gives us

$$\rho(u_1 \vee v \vee w) \leq \rho(x \vee y_1 \vee w).$$

The definitions of u_1 , y_1 , and w , imply that $u_1 \vee v \vee w = u \vee v$ and $x \vee y_1 \vee w = x \vee y$, and the desired inequality follows. Finally, use the statement about strict inequalities in Lemma 4.1 to justify the statement about strict inequalities in the conclusion of the Theorem. \blacksquare

COROLLARY 4.1. *If $x \wedge y = 0$, then $\rho(x) + \rho(y)$ can be defined unambiguously as $\rho(x \vee y)$.*

Addition is not everywhere-defined:

PROPOSITION 4.1. *For $\zeta, \eta \in \mathcal{R}$, the sum $\zeta + \eta$ exists only if $\zeta \leq \sim \eta$, or, equivalently, $\eta \leq \sim \zeta$.*

Proof. If $\zeta + \eta$ exists then for some $x, y \in \mathbb{A}$ we have $\rho(x) = \zeta$, $\rho(y) = \eta$, and $x \wedge y = 0$. But $x \wedge y = 0$ is equivalent to $x \leq \sim y$, and that implies $\zeta = \rho(x) \leq \rho(\sim y) = \sim \eta$. \blacksquare

A sum $\zeta + \eta$ may be undefined even when $\zeta \leq \sim \eta$, simply because there are no two elements $x, y \in \mathbb{A}$ such that $x \wedge y = 0$, $\rho(x) = \zeta$, and $\rho(y) = \eta$. The remedy to this unpleasant situation is in §5. The problem occurs in Example 3.6, where $\alpha \leq \sim \beta$, but $\alpha + \beta$ is nonetheless undefined. The addition table for that example appears in Figure 4. A “ \boxtimes ” marks the

$+$	0	α	β	γ	δ	1
0	0	α	β	γ	δ	1
α	α	β	\boxtimes	δ	1	\boxtimes
β	β	\boxtimes	\boxtimes	1	\boxtimes	\boxtimes
γ	γ	δ	1	\boxtimes	\boxtimes	\boxtimes
δ	δ	1	\boxtimes	\boxtimes	\boxtimes	\boxtimes
1	1	\boxtimes	\boxtimes	\boxtimes	\boxtimes	\boxtimes

FIG. 4. Addition table for Example 3.6.

places where Proposition 4.1 explains why the entry is undefined. A “ \boxtimes ” marks the other places where the entry is undefined. Example 3.4 is a scale for which this particular pathology – that $\zeta + \eta$ may be undefined even though $\zeta \leq \sim \eta$ – never occurs.

Some ways in which addition is obviously well-behaved are these: For $\zeta, \eta \in \mathcal{R}$ we have $\zeta + 0 = \zeta$, $\zeta + \sim \zeta = 1$, and $\zeta + \eta = \eta + \zeta$, the existence of either of these sums entailing that of the other.

What about associativity? If $\zeta + (\eta + \theta)$ exists then some $y, z \in \mathbb{A}$ whose images under ρ are η and θ respectively, satisfy $y \wedge z = 0$, and some $x, w \in \mathbb{A}$ whose images under ρ are ζ and $\eta + \theta = \rho(y \vee z)$ satisfy $x \wedge w = 0$. Neither y nor z was assumed disjoint from x . Can w be split into disjoint parts whose images under ρ are those of y and z ? Not always. When $\zeta + (\eta + \theta)$ and $(\zeta + \eta) + \theta$ both exist are they always equal? A partial answer is obvious:

PROPOSITION 4.2. *If $x_1, \dots, x_n \in \mathbb{A}$ are pairwise disjoint and $\rho(x_i) = \zeta_i$ for $i = 1, \dots, n$, then $\zeta_1 + \dots + \zeta_n$ is unambiguously defined.*

4.2. Duality, modularity, subtraction, relative complementation, and de Morganism

4.2.1. Duality

The array (2) of identities on page 6 defining the concept of Boolean algebra has an evident symmetry: Interchange the roles of “ \wedge ” and “ \vee ” and

of “0” and “1”, and the identities in that table are merely permuted among themselves. If the partial ordering \leq on a Boolean algebra is regarded as part of the structure, interchange it with its inverse \geq . All consequences of those identities then remain true if this same interchange of relations and operations is applied to them. The interchange leaves the operation of complementation unchanged, i.e., that operation is self-dual. That much is well-known. The same thing applies not only to Boolean algebras but also to scales generally. In particular, the dual of Theorem 4.1 says:

If $x \vee y = 1$ then $\rho(x \wedge y)$ depends on x and y only through $\rho(x)$ and $\rho(y)$, and in a strictly increasing fashion.

Therefore a “dual-addition” is unambiguously defined. We shall call the values of this operation “dual-sums” and write

$$\rho(x \wedge y) = \rho(x) \oplus \rho(y) \text{ if } x \vee y = 1.$$

The dual of Proposition 4.1 says that $\zeta \oplus \eta$ exists only if $\zeta \geq \sim \eta$, or equivalently, $\eta \geq \sim \zeta$. The dual-addition table for Example 3.6 on page 14 is constructed by first reflecting the interior, but not the margins, of the table in Figure 4 about the diagonal that contains only 1’s, and then replacing each entry in the interior, but not in the margins, by its complement: $0 \leftrightarrow 1, \alpha \leftrightarrow \delta, \beta \leftrightarrow \gamma, \boxtimes \leftrightarrow \boxtimes, \boxed{?} \leftrightarrow \boxed{?}$.

4.2.2. Modularity

The following “modular law” is the essential tool for dealing with subtraction, cancellation, and relative complementation.

LEMMA 4.2. *If $x \wedge y = 0$ and $y \vee z = 1$ then no ambiguity comes from writing*

$$\rho(x) + \rho(y) \oplus \rho(z).$$

In other words, we have $\{\rho(x) + \rho(y)\} \oplus \rho(z) = \rho(x) + \{\rho(y) \oplus \rho(z)\}$. In particular, the sums and dual-sums exist.

Proof. Since $x \wedge y = 0$ we also have $x \wedge (y \wedge z) = 0$ so $\rho(x) + \{\rho(y) \oplus \rho(z)\}$ exists and is equal to $\rho(x \vee (y \wedge z))$. Since $y \vee z = 1$ we have $(x \vee y) \vee z = 1$, so $\{\rho(x) + \rho(y)\} \oplus \rho(z)$ exists and is equal to $\rho((x \vee y) \wedge z)$. Finally, the two identities $x \wedge y = 0$ and $y \vee z = 1$ entail that $x \vee (y \wedge z) = (x \vee y) \wedge z$. ■

4.2.3. Subtraction

PROPOSITION 4.3. *The functions*

$$\zeta \mapsto \zeta + \eta \quad \text{and} \quad \zeta \mapsto \zeta \oplus \sim \eta$$

are inverses. In particular, the not-everywhere-defined nature of the operations involved does not prevent the image of each of these functions from coinciding with the domain of the other.

In other words, subtraction of η from ζ yields $\zeta - \eta = \zeta \oplus \sim \eta$.

Proof. Suppose $x \wedge y = 0$, $\rho(x) = \zeta$, and $\rho(y) = \eta$. Since $y \vee \sim y = 1$, Lemma 4.2 (the modular law) applies:

$$(\zeta + \eta) \oplus \sim \eta = \zeta + (\eta \oplus \sim \eta) = \zeta + 0 = \zeta.$$

So the second function is a left-inverse of the first. To prove the first is a left-inverse of the second, dualize, interchanging “ \wedge ” with “ \vee ”, “0” with “1”, and “+” with “ \oplus ”. ■

PROPOSITION 4.4. *The difference $\zeta - \eta$ exists only if $\zeta \geq \eta$.*

Proof. The dual of Proposition 4.1 implies $\zeta \oplus \sim \eta$ exists only if $\zeta \geq \eta$. ■

But $\zeta - \eta$ is sometimes undefined even when $\zeta \geq \eta$. In Example 3.6, we have $\gamma > \alpha$, but no members x, y of the domain simultaneously satisfy $y > x$, $\rho(y) = \gamma$, and $\rho(x) = \alpha$. Thus we cannot subtract α from γ . This difficulty will be remedied in §5.

4.2.4. Relative complementation

Lemma 4.2 (the modular law) can be used to prove that a scaling induces an operation of additive relative complementation on its image. Like addition and subtraction, this is not everywhere-defined.

PROPOSITION 4.5. *If $x \in [a, b]$ then $\rho(\sim x_{[a,b]})$ depends on x , a , and b only through $\rho(x)$, $\rho(a)$, and $\rho(b)$.*

Recall the difficulty: In effect the proposition says if $x \in [a, b]$, $y \in [c, d]$, and $\rho(a) = \rho(c) \leq \rho(x) = \rho(y) \leq \rho(b) = \rho(d)$ then $\rho(\sim x_{[a,b]}) = \rho(\sim y_{[c,d]})$. But only in case $[a, b] = [c, d]$ is this immediate from Definition 2.3.

Proof. Since $a \leq x \leq b$ we have $a \wedge \sim x = 0$ and $(\sim x) \vee b = 1$. Therefore, by Lemma 4.2 (the modular law), the following is unambiguously defined.

$$\rho(\sim x_{[a,b]}) = \rho(a) + \rho(\sim x) \oplus \rho(b).$$

■

Example 3.6 shows why additive relative complementation on a scale is not everywhere-defined. In that example α is its own additive complement relative to the interval $[0, \beta]$. But the additive complement of α relative to the interval $[0, \gamma]$ does not exist even though $0 < \alpha < \gamma$, because there do not exist $0 < x < y$ in the domain of ρ whose respective images under ρ are $0, \alpha$, and γ . This will be remedied in §5.

In §3 we remarked that Example 3.8 “provide[s] us with an example of something that ‘ought to be’ a relative complement but is not.” In that example we have $\alpha < \gamma < \varepsilon$ and $\beta < \gamma < \delta$, but no complements $\sim\gamma_{[\alpha, \varepsilon]}$ or $\sim\gamma_{[\beta, \delta]}$ exist, even though $\alpha + (\sim\gamma) \oplus \varepsilon = \gamma$ and $\beta + (\sim\gamma) \oplus \delta = \gamma$ do exist.

The operation of additive relative complementation on a scale depends not only on the partial ordering of the scale but also on the scaling. This can be seen by considering Example 3.9. The additive complement of $\{a, b\}$ relative to the interval from $\{a\}$ to $\{a, b, c\}$ is $\{a, c\}$, not only in the domain, but also in the range! But nothing in the partial ordering of that scale makes $\{a, c\}$ a better candidate than $\{b, c\}$ to be the additive relative complement. Rather, it is singled out by the ordering of the domain.

4.2.5. *de Morganism*

The next proposition is immediate from the results of this section and will be useful in §5:

PROPOSITION 4.6.

If $\zeta + \eta$ exists, then so does $(\sim\zeta) \oplus (\sim\eta)$, and $\sim(\zeta + \eta) = (\sim\zeta) \oplus (\sim\eta)$.

If $\zeta \oplus \eta$ exists, then so does $(\sim\zeta) + (\sim\eta)$, and $\sim(\zeta \oplus \eta) = (\sim\zeta) + (\sim\eta)$.

4.2.6. *Technical lemma on inequalities*

The following lemma will be useful in §5.

LEMMA 4.3.

$$\text{If } \zeta \leq \eta \text{ and } \zeta + \theta \text{ and } \eta + \theta \text{ exist, then } \zeta + \theta \leq \eta + \theta, \quad (4)$$

$$\text{If } \zeta \leq \eta \text{ and } \zeta \oplus \theta \text{ and } \eta \oplus \theta \text{ exist, then } \zeta \oplus \theta \leq \eta \oplus \theta, \quad (5)$$

$$\text{If } \zeta \leq \eta \text{ and } \zeta - \theta \text{ and } \eta - \theta \text{ exist, then } \zeta - \theta \leq \eta - \theta, \quad (6)$$

and all three statements remain true if “ $<$ ” replaces both occurrences of “ \leq ”.

Proof. (4) follows from the conjunction of Theorem 4.1 with the definition embodied in Corollary 4.1. (5) is the dual of (4). (6) follows from the conjunction of (5) with Proposition 4.3 and the strictly decreasing nature of (absolute) complementation. ■

4.3. Measures on scales

DEFINITION 4.1. A **finitely additive measure** on a scale \mathcal{R} is a strictly increasing mapping $\mu : \mathcal{R} \rightarrow [0, \infty) \subseteq \mathbb{R}$ satisfying

1. For $\zeta \in \mathcal{R}$, if $\zeta > 0$ then $\mu(\zeta) > 0$, and
2. For $\zeta, \eta \in \mathcal{R}$, if $\zeta + \eta$ exists then $\mu(\zeta + \eta) = \mu(\zeta) + \mu(\eta)$.

A scale is **measurable** if it is the domain of a measure.

The words “strictly increasing” would be redundant if the domain \mathcal{R} were a Boolean algebra. They are also redundant in what we shall call “divided” scales – to be defined in the next section. That they are not redundant in this more general setting is shown by this example: Let ρ be defined on the set of all four subsets of $\{a, b\}$ and suppose $0 < \rho(\{a\}) < \rho(\{b\}) < 1$. Let $0 = \mu(0) < \mu(\rho(\{a\})) = 2/3 \not\leq 1/3 = \mu(\rho(\{b\})) < \mu(1) = 1$.

Clearly this generalizes Definition 2.7. Moreover, if $\mu : \mathcal{R} \rightarrow [0, \infty)$ is a measure and $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is the basic scaling that induces the scale-structure on \mathcal{R} , then $\mu \circ \rho : \mathbb{A} \rightarrow [0, \infty)$ is a measure on the Boolean algebra \mathbb{A} .

5. DIVISIBILITY AND MEASURABILITY

5.1. Dividedness

In § 4 we saw three pathologies:

1. Although $\zeta \leq \sim \eta$, or equivalently $\eta \leq \sim \zeta$, is necessary for the existence of $\zeta + \eta$, in some scales it is not sufficient because there may be no $x, y \in \mathbb{A}$ for which $x \wedge y = 0$ and $\rho(x) = \zeta$ and $\rho(y) = \eta$;
2. Although $\zeta \geq \eta$ is necessary of the existence of $\zeta - \eta$, in some scales it is not sufficient because there may be no $x, y \in \mathbb{A}$ for which $x > y$ and $\rho(x) = \zeta$ and $\rho(y) = \eta$;
3. Although $\zeta \leq \eta \leq \theta$ is necessary for the existence of $\sim \eta_{[\zeta, \theta]}$, in some scales it is not sufficient because there may be no $x, y, z \in \mathbb{A}$ for which $x \leq y \leq z$ and $\rho(x) = \zeta$, $\rho(y) = \eta$, and $\rho(z) = \theta$.

PROPOSITION 5.1. *These three pathologies are equivalent, i.e., in any scale in which one of them occurs, so do the others.*

Proof. The sum $\zeta + \eta$ and the difference $(\sim\eta) - \zeta = (\sim\eta) \oplus (\sim\zeta)$ are complements of each other, and complementation on a scale is a bijection. This suffices for equivalence of (1) and (2). Existence of the relative complement $\sim\eta_{[\zeta, \theta]}$ is equivalent to the existence of both the sum $\zeta + (\sim\eta)$ and the difference $\theta - \eta$. This suffices for equivalence of (3) with its predecessors. ■

A simple law additional to those that define a scale is the remedy.

DEFINITION 5.1. A basic scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is **divided** if whenever $\rho(x) < \rho(y)$ then some $y_1 < y$ satisfies $\rho(y_1) = \rho(x)$. The domain \mathbb{A} and the range \mathcal{R} will also be called “divided” if ρ is divided.

In case \mathbb{A} is an algebra of subsets of a set and ρ is a measure, this says the measurable set y is the union of smaller sets, one of which has the same measure as x . The reader can check that Examples 3.1, 3.4, and 3.5 are divided, and Example 3.6 is not divided.

THEOREM 5.1. *If a scale \mathcal{R} is divided, then for $\zeta, \eta, \theta \in \mathcal{R}$,*

1. $\zeta + \eta$ exists if $\zeta \leq \sim\eta$ (or equivalently, if $\eta \leq \sim\zeta$).
2. $\zeta - \eta$ exists if $\zeta \geq \eta$.
3. $\sim\eta_{[\zeta, \theta]}$ exists if $\zeta \leq \eta \leq \theta$.

Proof. Item (2) follows immediately from the definition of dividedness.

It follows that under the assumptions of item (1), (i.e., that $\zeta \leq \sim\eta$), ζ can be subtracted from $\sim\eta$. Then we have $(\sim\eta) - \zeta = (\sim\eta) \oplus (\sim\zeta)$; in particular, this latter dual-sum exists. By Proposition 4.6, on “de-Morganism,” so does the sum $\zeta + \eta$.

Item (3) follows from the conjunction of items (1) and (2) and the observation that $\sim\eta_{[\zeta, \theta]} = \zeta + (\theta - \eta)$. (Note that the parentheses in “ $\zeta + (\theta - \eta)$ ” need to be where they are.) ■

5.2. Atoms

DEFINITION 5.2.

1. An element $x \neq 0$ of a Boolean algebra \mathbb{A} is an **atom** of \mathbb{A} if the interval $[0, x]$ contains only 0 and x .
2. A Boolean algebra \mathbb{A} is **atomic** if for every $y \in \mathbb{A}$ there is some atom $x \in \mathbb{A}$ such that $x \leq y$.
3. A Boolean algebra \mathbb{A} is **atomless** if it contains no atoms.

EXAMPLE 5.1. The Boolean algebra of all subsets of a set is atomic. Each singleton, i.e., each subset with only one member, is an atom.

EXAMPLE 5.2. The Boolean algebra of all clopen (i.e., simultaneously closed and open) subsets of the Cantor set is atomless.

EXAMPLE 5.3. Adjoin a finite set of isolated points to the Cantor set. The Boolean algebra of all clopen subsets of the resulting space is neither atomic nor atomless. The singleton of each isolated point is an atom.

5.3. Divisions

§5.1 may appear to be suggesting that in cases in which $\zeta + \eta$ does not exist, we should seek some larger Boolean algebra and a correspondingly larger scale in which the sum $\zeta + \eta$ will be found. Hence we have the following definition.

DEFINITION 5.3. A **division** of a basic scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is a scaling $\rho_1 : \mathbb{A}_1 \rightarrow \mathcal{R}_1$ such that

1. ρ_1 is divided;
2. \mathbb{A} is a subalgebra of \mathbb{A}_1 ;
3. \mathcal{R} is a sub-poset of \mathcal{R}_1 ;
4. ρ is the restriction of ρ_1 to \mathcal{R} ; and
5. No pair intermediate between $(\mathbb{A}, \mathcal{R})$ and $(\mathbb{A}_1, \mathcal{R}_1)$ satisfies 1-4.

The domain \mathbb{A}_1 and the range \mathcal{R}_1 of ρ_1 will also be called “divisions” of \mathbb{A} and \mathcal{R} respectively.

EXAMPLE 5.4. Regard the Boolean algebra \mathbb{A} of all four subsets of $\{a, b\}$ as a subalgebra of the Boolean algebra \mathbb{A}_1 of all subsets of $\{a, b_1, b_2\}$ by identifying b with $\{b_1, b_2\}$, so that the atom b has been split. Then the scale on the right in Figure 5 is a division of the one on the left.

EXAMPLE 5.5. Regard the Boolean algebra \mathbb{A} of all eight subsets of $\{a, b, c\}$ as a subalgebra of the Boolean algebra \mathbb{A}_1 of all subsets of $\{a, b, c_1, c_2\}$ by identifying c with $\{c_1, c_2\}$, so that the atom c has been split. A division of the scale in Figure 6 appears in Figure 7.

EXAMPLE 5.6. Suppose \mathbb{A} is the Boolean algebra of all subsets of $\{a, b, c, d, e\}$, and ρ is the scaling arising in the manner described in Example 3.3 from the convex set \mathcal{C} of all measures m on \mathbb{A} that satisfy $m(\{a, b, c\}) < m(\{d, e\})$. Then $\rho(\{a, b, c\}) < \rho(\{d, e\})$, and the only other sets $S, T \in \mathbb{A}$ for which $\rho(S) \leq \rho(T)$ are those for which $S \subseteq T$. Split a into disjoint parts a_d and a_e , similarly b into b_d and b_e , and c into c_d and c_e . Split d into four disjoint parts $d_a, d_b, d_c, d_{\text{ceterus}}$, and e into $e_a, e_b, e_c, e_{\text{ceterus}}$. The convex set \mathcal{C} is then naturally identified with the set of all measures m_1 on the Boolean algebra \mathbb{A}_1 of all subsets of

$$\{a_d, a_e, b_d, b_e, c_d, c_e, d_a, d_b, d_c, d_{\text{ceterus}}, e_a, e_b, e_c, e_{\text{ceterus}}\}$$

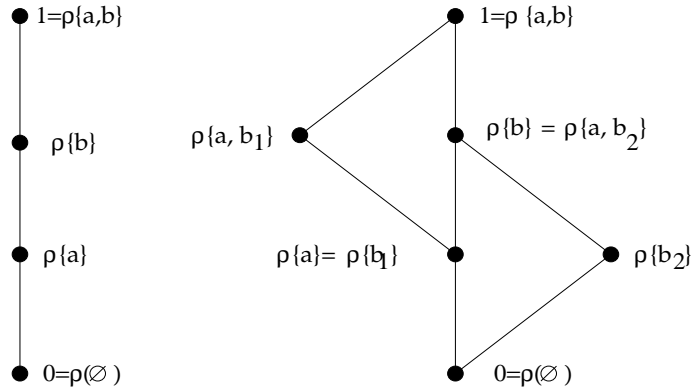


FIG. 5.

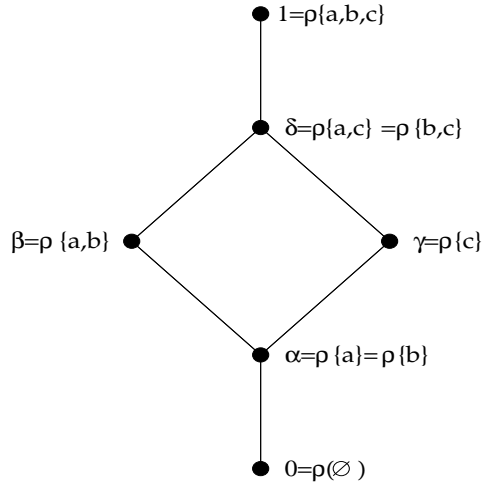


FIG. 6.

that satisfy

$$m_1(\{ a_d, a_e, b_d, b_e, c_d, c_e \}) < m_1(\{ d_a, d_b, d_c, d_{ceterus}, e_a, e_b, e_c, e_{ceterus} \}).$$

Let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the smaller class of measures m_1 that satisfy this inequality and also $m_1(\{ a_d \}) = m_1(\{ d_a \})$, $m_1(\{ a_e \}) = m_1(\{ e_a \})$, and so on. (If we had had $\rho(\{ a, b, c \}) = \rho(\{ d, e \})$, i.e., “=” instead of “<”, then we would have omitted $d_{ceterus}$ and $e_{ceterus}$, which are slack components.) Then the scaling ρ_1 arising from \mathcal{C}_1 in the manner of Example 3.3 is a division of ρ .

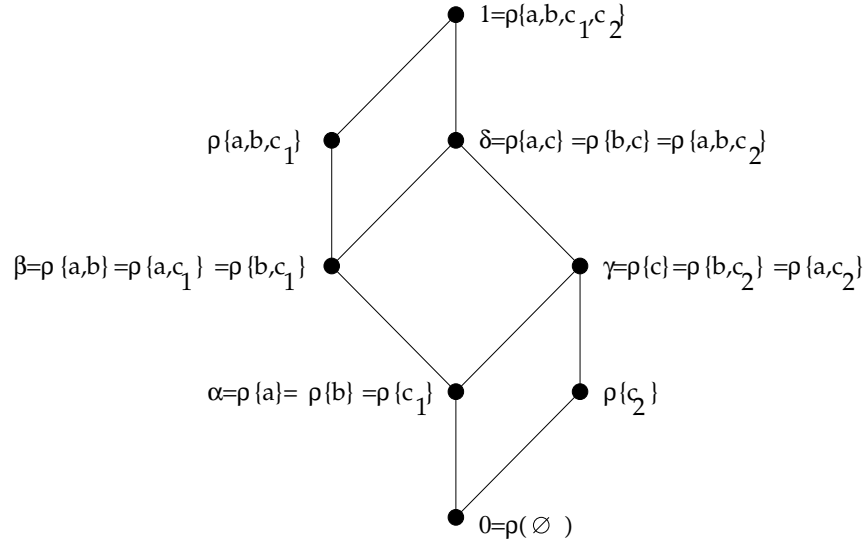


FIG. 7.

5.4. Divisibility

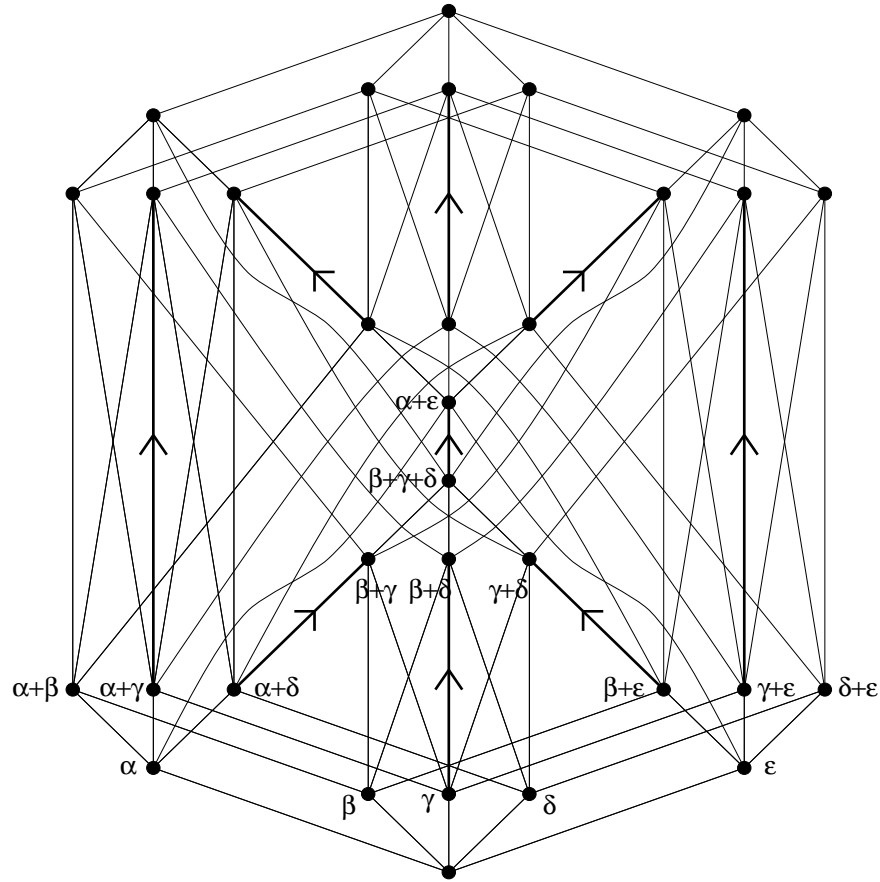
5.4.1. Defined

Does every finite scale have a division? The next definition foreshadows the answer.

DEFINITION 5.4. A scale is **divisible** if it has a division; otherwise it is **indivisible**.

5.4.2. Adaptation of the Kraft-Pratt-Seidenberg counterexample

EXAMPLE 5.7. Figure 8 depicts my adaptation of an object constructed by Kraft, Pratt, and Seidenberg in [10]. Their purpose was to exhibit counterexample to Bruno de Finetti's conjecture in [2] that every *linear* ordering of a finite Boolean algebra of propositions by comparative probabilities that satisfies the assumptions (1) of §1.6 (including weak additivity) has an "agreeing measure." A broad generalization of that conjecture states, in the language of the present paper, that every finite scale is measurable. This adaptation is a scale that is neither measurable nor divisible (and so is the linearly ordered example of which it is an adaptation).



Adaptation of the Kraft-Pratt-Seidenberg counterexample:
A non-measurable and indivisible scale.

FIG. 8.

This scale is the image of the Boolean algebra of all subsets of $\{a, b, c, d, e\}$ under a one-to-one scaling ρ satisfying $\rho(\{a\}) = \alpha, \rho(\{b\}) = \beta, \rho(\{c\}) = \gamma, \rho(\{d\}) = \delta, \rho(\{e\}) = \varepsilon$. The thin lines in Figure 8 correspond to subset relations, e.g., a thin line goes from $\rho(\{a\}) = \alpha$ to $\rho(\{a, e\}) = \alpha + \varepsilon$ because $\{a\} \subseteq \{a, e\}$. The thick lines with arrows on them correspond to

the following inequalities not necessitated by subset relations.

$$\begin{aligned}
\alpha + \delta &< \beta + \gamma \\
\beta + \varepsilon &< \gamma + \delta \\
\gamma &< \beta + \delta \\
\beta + \gamma + \delta &< \alpha + \varepsilon
\end{aligned} \tag{7}$$

PROPOSITION 5.2. *Example 5.7 is a non-measurable scale.*

Although the Proposition is new, the proof is due to Kraft, Pratt, and Seidenberg.

Proof. Suppose μ is a measure on this scale. Then from (7) it follows that

$$\begin{aligned}
\mu(\alpha) + \mu(\delta) &< \mu(\beta) + \mu(\gamma) \\
\mu(\beta) + \mu(\varepsilon) &< \mu(\gamma) + \mu(\delta) \\
\mu(\gamma) &< \mu(\beta) + \mu(\delta) \\
\mu(\beta) + \mu(\gamma) + \mu(\delta) &< \mu(\alpha) + \mu(\varepsilon).
\end{aligned} \tag{8}$$

Unlike the addition in (7), this is old-fashioned everywhere-defined and impeccably-behaved addition of real numbers. Therefore we can deduce an inequality between the sum of the four left sides and that of the four right sides, an absurdity:

$$\mu(\alpha) + 2\mu(\beta) + 2\mu(\gamma) + 2\mu(\delta) + \mu(\varepsilon) < \mu(\alpha) + 2\mu(\beta) + 2\mu(\gamma) + 2\mu(\delta) + \mu(\varepsilon).$$

■

PROPOSITION 5.3. *Example 5.7 is an indivisible scale.*

Proof. Observe that all sums in this proof are sums of images of pairwise disjoint members of a division \mathbb{A}_1 of \mathbb{A} , so by Proposition 4.2 we have all the associativity we need. In particular, we will deal with two members of \mathbb{A}_1 whose images under ρ are equal to γ . They are disjoint.

Assume it is divisible, so that Theorem 5.1 is applicable. Since $\gamma < \beta + \gamma + \delta < \alpha + \varepsilon = \sim(\beta + \gamma + \delta)$, it follows from Theorem 5.1 (1) that the sum $\gamma + (\beta + \gamma + \delta)$ exists. (If we didn't have Theorem 5.1, we would use a more leisurely but essentially equivalent approach: Split the set $\{a, e\}$ according to the manner of Example 5.6 so that one of its subsets after splitting has $\beta + \gamma + \delta$ as its image under ρ , and that subset is disjoint for $\{c\}$, so the addition can be done. That is how Theorem 5.1 saves us some

work.) Now Lemma 4.3 can be applied, and we can add the first two of the inequalities (7) to get

$$\alpha + \beta + \delta + \varepsilon < \beta + \gamma + \gamma + \delta.$$

Then Theorem 5.1 (2) implies that we can subtract $\beta + \delta$ from both sides of this inequality, getting

$$\alpha + \varepsilon < \gamma + \gamma.$$

Since (7) tells us that $\beta + \gamma + \delta < \alpha + \varepsilon$, it follows that

$$\beta + \gamma + \delta < \gamma + \gamma.$$

Theorem 5.1 (2) tells us we can subtract γ from both sides of this, getting

$$\beta + \delta < \gamma.$$

This contradicts the inequalities (7). ■

5.5. Divisibility and measurability

In this section, as in §3 we denote the cardinality of a set T by $|T|$.

LEMMA 5.1. *Suppose $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is a divided basic scaling and \mathbb{A} is the Boolean algebra of all subsets of a finite set S . Then for $T_1, T_2 \subseteq S$, if $\rho(T_1) = \rho(T_2)$ then $|T_1| = |T_2|$, and if $\rho(T_1) < \rho(T_2)$ then $|T_1| < |T_2|$. (The converse is false since $\rho(T_1)$ and $\rho(T_2)$ can be incomparable.)*

Proof. First assume $\rho(T_1) = \rho(T_2)$. Proceed by induction on $|T_1|$. If $|T_1| = 0$ then the conclusion follows from the strictly increasing nature of ρ . Then suppose $|T_1| = n + 1$ and $\rho(T_1) = \rho(T_2)$. For some $t \in T_1$, dividedness implies that $\rho(T_1 \setminus \{t\})$ must be the same as the image under ρ of some proper subset of T_2 . Then apply the induction hypothesis.

If $\rho(T_1) < \rho(T_2)$ then, by divideness, we can find $T'_2 \subsetneq T_2$ such that $\rho(T_1) = \rho(T'_2)$. Then proceed as above with T'_2 in place of T_2 . ■

THEOREM 5.2. *Suppose $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is a divided basic scaling on the Boolean algebra \mathbb{A} of all subsets of a finite set S . Then there is a partition S_1, \dots, S_k of S such that for every $T_1, T_2 \subseteq S$, $\rho(T_1) \leq \rho(T_2)$ if and only if $|T_1 \cap S_i| \leq |T_2 \cap S_i|$ for $i = 1, \dots, k$.*

In effect this says members of \mathcal{R} can be represented as k -tuples (t_1, \dots, t_k) , and the i th component t_i counts the number of members of a set that are

in the i th equivalence class, and moreover $(t_1, \dots, t_k) \leq (u_1, \dots, u_k)$ precisely if $t_i \leq u_i$ for every i . The largest member of \mathcal{R} would correspond to (a_1, \dots, a_k) where each a_i is the whole number of members of the i th equivalence class.

Proof. For two members $s, t \in S$, Lemma 5.1 implies that $\rho(\{s\})$ and $\rho(\{t\})$ are either equal or incomparable. Call s and t equivalent iff $\rho(\{s\}) = \rho(\{t\})$, and call the equivalence classes S_1, \dots, S_k . If T_1, T_2 are both subsets of the same equivalence class then by additivity we have $\rho(T_1) = \rho(T_2)$ or $\rho(T_1) < \rho(T_2)$ according as $|T_1| = |T_2|$ or $|T_1| < |T_2|$. More generally, additivity implies that if $|T_1 \cap S_i| \leq |T_2 \cap S_i|$ for $i = 1, \dots, k$ then $\rho(T_1) \leq \rho(T_2)$, with equality between the two values of ρ if and only if equality holds between the two cardinalities for every $i \in \{1, \dots, k\}$.

Next we need to show that if for some $i, j \in \{1, \dots, k\}$ we have $|T_1 \cap S_i| < |T_2 \cap S_i|$ and $|T_2 \cap S_j| > |T_1 \cap S_j|$ then $\rho(T_1)$ and $\rho(T_2)$ are incomparable. To see this, first create $U_1, U_2 \subseteq S$ as follows. For each $i \in \{1, \dots, k\}$ for which $|T_1 \cap S_i| < |T_2 \cap S_i|$, delete $|T_1 \cap S_i|$ members from $T_2 \cap S_i$, including, but not limited to, all members of $T_1 \cap T_2 \cap S_i$, to get $U_2 \cap S_i$, so that U_2 is the union of all k of these intersections. Similarly, for each $j \in \{1, \dots, k\}$ for which $|T_1 \cap S_j| > |T_2 \cap S_j|$, delete $|T_2 \cap S_j|$ members from $T_1 \cap S_j$, including, but not limited to, all members of $T_1 \cap T_2 \cap S_i$, to get $U_1 \cap S_j$, so that U_1 is the union of all k of these intersections. Then for each $i \in \{1, \dots, k\}$ for which $|T_1 \cap S_i| < |T_2 \cap S_i|$, we have in effect deleted *all* of the members of $T_1 \cap S_i$ from T_1 , getting $U_1 \cap S_1 = \emptyset$, and we have deleted the same number of members of $T_2 \cap S_i$ from T_2 to get $U_2 \cap S_i = \emptyset$. Since all members of S_i have the same image under ρ , and since in divided scales we can subtract, we have subtracted the same thing from both sides of either the equality $\rho(T_1) = \rho(T_2)$ or the inequality $\rho(T_1) < \rho(T_2)$. Therefore we must have $\rho(U_1) = \rho(U_2)$ or $\rho(U_1) < \rho(U_2)$, according as the equality or the inequality holds between $\rho(T_1)$ and $\rho(T_2)$. And there is no $i \in \{1, \dots, k\}$ for which S_i intersects both U_1 and U_2 . If $u_1 \in U_1$ then, by divisibility, there exists $U'_2 \subseteq U_2$ such that $\rho(\{u_1\}) = \rho(U'_2)$. By Lemma 5.1 this implies U'_2 has only one member — call it u_2 . But then $\rho(\{u_1\}) = \rho(\{u_2\})$ even though u_1 and u_2 are in different equivalence classes — a contradiction following from the assumption of comparability of $\rho(T_1)$ and $\rho(T_2)$. ■

In effect we have proved that, under the assumptions of the theorem, \mathcal{R} must be a finite “Kleene algebra.” This concept generalizes the concept of Boolean algebra. A Kleene algebra is a bounded distributive lattice with a

certain sort of complementation, which is *not* a “complementation” as the term is understood in lattice theory. The precise definition is: A Kleene algebra is a partially ordered set with largest and smallest members 1 and 0 (this is “boundedness”) in which any set $\{x, y\}$ of two members has an infimum $x \vee y$ and a supremum $x \wedge y$ (i.e., it is a lattice), and these two operations distribute over each other, and there is an complementation $x \mapsto \sim x$ satisfying:

$$\begin{aligned} \sim 0 &= 1, \\ \sim \sim x &= x, \\ \sim(x \wedge y) &= (\sim x) \vee (\sim y), \\ x \wedge \sim x &\leq y \vee \sim y. \end{aligned}$$

In a Boolean algebra we would have $x \wedge \sim x = 0$ and $y \vee \sim y = 1$ (i.e., this would be a lattice-theoretic complementation) instead of this weaker last condition. A Boolean algebra can be defined as a “complemented distributive lattice.” Up to isomorphism, a finite Kleene algebra is the same thing as a family of sub-multisets of a finite multiset, that is closed under the three operations.

NOTATIONAL CONVENTION. In order to use it as a tool in the statement and proof of the next theorem, we further develop the notation introduced in the paragraph after Theorem 5.2. For any divided basic scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ on a finite Boolean algebra \mathbb{A} , we represent members of \mathcal{R} as tuples (t_1, \dots, t_k) of non-negative integers. For any two such k -tuples (t_1, \dots, t_k) and (u_1, \dots, u_k) , we have $(t_1, \dots, t_k) \leq (u_1, \dots, u_k)$ iff $t_i \leq u_i$ for $i = 1, \dots, k$. Addition and subtraction of members of \mathcal{R} then become term-by-term addition and subtraction of components. If $\alpha = (t_1, \dots, t_k)$, $\beta = (u_1, \dots, u_k)$, $\gamma = (v_1, \dots, v_k)$, and $\alpha \leq \beta \leq \gamma$, then the additive relative complement $\sim \beta_{[\alpha, \gamma]}$ is $(v_1 - u_1 + t_1, \dots, v_k - u_k + t_k)$. The range \mathcal{R} also has a lattice structure. (Recall from Example 3.4 that a divided scale need not be a lattice if it is not finite, and from Example 3.9 that a finite scale need not be a lattice if it is not divided.) The lattice structure of a finite divided scale is given by the componentwise definition of the meet and join operations:

$$\begin{aligned} (t_1, \dots, t_k) \wedge (u_1, \dots, u_k) &= (t_1 \wedge u_1, \dots, t_k \wedge u_k) \\ (t_1, \dots, t_k) \vee (u_1, \dots, u_k) &= (t_1 \vee u_1, \dots, t_k \vee u_k). \end{aligned}$$

THEOREM 5.3. *Suppose $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is a basic scaling and \mathbb{A} is finite. Then \mathcal{R} is divisible if and only if it is measurable.*

Proof. If $\rho_1 : \mathbb{A}_1 \rightarrow \mathcal{R}_1$ is a division of $\rho : \mathbb{A} \rightarrow \mathcal{R}$, and $\mu_1 : \mathcal{R}_1 \rightarrow \mathbb{R}$ is a measure, then the restriction of μ_1 to \mathcal{R} is also a measure. Therefore no generality is lost by assuming the scale is not just divisible, but divided, and so we do. Since \mathbb{A} is finite, we lose no generality by assuming \mathbb{A} is the algebra of all subsets of some finite set S .

Following the notation introduced in the paragraph after Theorem 5.2, write $\rho(T) = (t_1, \dots, t_k)$ for $T \subseteq S$. For any $m_1, \dots, m_k > 0$, the function $\mu(T) = \sum_{i=1}^k m_i t_i$ is a measure of the sort required, and the set of all such measures is the requisite convex set of measures.

Conversely, assume \mathcal{R} is measurable. For every measure $\mu : \mathcal{R} \rightarrow \mathbb{R}$, the mapping $\mu \circ \rho : \mathbb{A} \rightarrow \mathbb{R}$ is a measure on the underlying Boolean algebra. The set of all such measures $\mu \circ \rho$ satisfying $\mu(\rho(1)) = 1$ is convex and bounded. Since it is finite, we may take \mathbb{A} to be the algebra of all subsets of a finite set S . For any $\zeta, \eta \in \mathcal{R}$ satisfying $\zeta < \eta$, and any $T_1, T_2 \subseteq S$ for which $\rho(T_1) = \zeta$ and $\rho(T_2) = \eta$ we have an inequality

$$\sum_{t \in T_1} \mu(\rho(\{t\})) < \sum_{t \in T_2} \mu(\rho(\{t\})). \quad (9)$$

We get finitely many such inequalities, plus one equality that says

$$\sum_{t \in S} \mu(\rho(\{t\})) = 1. \quad (10)$$

The solution set of the system consisting of the inequalities (9) and the equation (10) in the finitely many variables $\mu(\rho(\{t\}))$, $t \in S$, is a bounded convex set that is the convex hull of finitely many ‘‘corners,’’ and each corner is a rational point in \mathbb{R}^n , where $n = |S|$. Let m be the number of corners, and let M be the $m \times n$ matrix whose rows are the corners. For each corner c call the corresponding row of M the c^{th} row, and let d_c be the common denominator of the rational numbers that are the entries in the c^{th} row. Note that each column of M corresponds to one of the variables $\mu(\rho(\{t\}))$, and so each column of M corresponds to one of the members $t \in S$. Call that column the t^{th} column of M . Let D be the diagonal matrix whose entries are the d_c . Then $M^T D$ is an integer matrix. For each $t \in S$, let $\sigma(\{t\}) =$ the t^{th} row of $M^T D$. Then, following the NOTATIONAL CONVENTION that precedes the statement of the Theorem, σ is the desired divided scaling. ■

EXAMPLE 5.8. Figure 10 depicts the convex set of all measures on the scale in Figure 9. The corners are the rows of the matrix

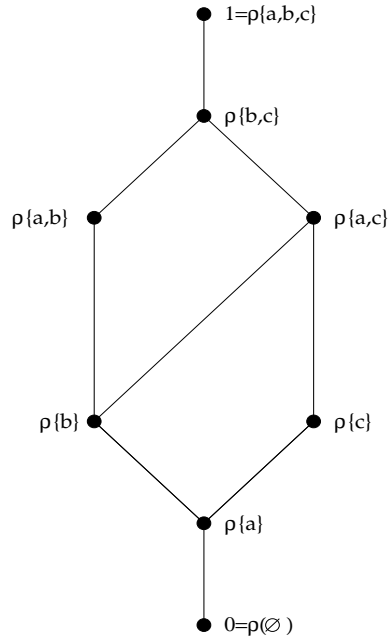


FIG. 9.

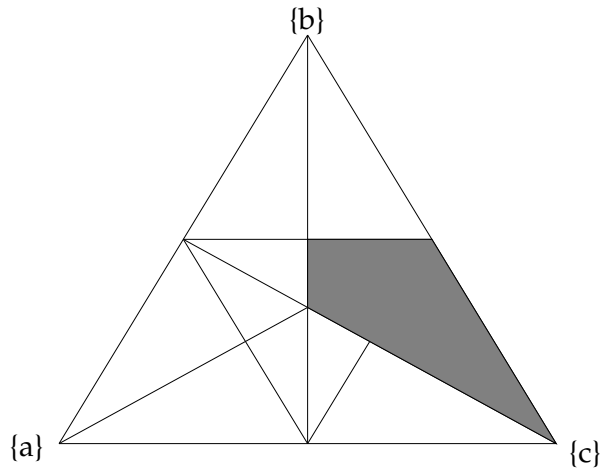


FIG. 10.

$$M = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The respective common denominators are 3, 4, 2, and 1, and we get

$$\begin{aligned} \sigma(\{a\}) &= (1, 1, 0, 0), \\ \sigma(\{b\}) &= (1, 2, 1, 0), \\ \sigma(\{c\}) &= (1, 1, 1, 1). \end{aligned}$$

This means: We split the atom a into two parts, and put one in the first equivalence class and one in the second; We split b into four parts, and put one in the first equivalence class, two in the second, and one in the third; We split c into four parts, and put one in each of the four equivalence classes. The first equivalence class has three members; the second has four; the third has two; the fourth has one.

5.6. Multiplication

Theorem 5.1 told us that if a scale is divided, then subtraction is generally defined, i.e., whenever $\zeta \leq \eta$ then $\eta - \zeta$ exists. Assume dividedness and Archimedeanism, but replace the assumption that $\zeta \leq \eta$, with the assumption that the scale is linearly ordered and $\zeta \neq 0$ (so, by Archimedeanism, ζ is not infinitesimal). Consider the maximum value of $n \in \{0, 1, 2, 3, \dots\}$ such that the following difference exists

$$\eta - \underbrace{\zeta - \zeta - \dots - \zeta}_n.$$

That maximum value may be 0, and must be finite. We must have

$$\alpha = \eta - \underbrace{\zeta - \zeta - \dots - \zeta}_n < \zeta$$

since otherwise we could subtract another ζ , contradicting the maximality of n (here we have used the assumption of linear ordering). Since $\alpha < \zeta$,

we can subtract α from ζ . Consequently we have an algorithm:

- (1) Let $i = 1$ (the positive integer 1, not the maximum element of a scale).
- (2) Let $\alpha = \zeta$.
- (3) Let $\beta = \eta$.
- (4) Let $n_i = \max \{n : \underbrace{\beta - \alpha - \dots - \alpha}_n \text{ exists}\}$
- (5) Let $\beta = \underbrace{\beta - \alpha - \dots - \alpha}_{n_i}$
- (6) If $\beta = 0$ then stop, else $\left\{ \begin{array}{l} \text{Increment } i \text{ to } i + 1; \\ \text{Interchange the values of } \alpha \text{ and } \beta; \\ \text{Go to (4).} \end{array} \right\}$.

If we had defined any reasonable notions of multiplication and division of members of a scale, then this algorithm would find the continued fraction expansion:

$$\frac{\eta}{\zeta} = n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \dots}}}$$

Call this the formal continued fraction expansion of the formal quotient η/ζ . Observe that the formal continued fraction expansions of the formal quotients η/ζ and θ/ζ are the same only if the difference between η and θ is infinitesimal, and therefore, by Archimedeanism, is 0. All this is summarized by a lemma:

LEMMA 5.2. *In a linearly ordered divided Archimedean scale, the continued fraction and the (non-zero) value of the denominator of a formal quotient determine the value of the numerator.*

THEOREM 5.4. *On any linearly ordered divided Archimedean scale \mathcal{R} there is exactly one measure $\mu : \mathcal{R} \rightarrow [0, 1] \subseteq \mathbb{R}$ such that $\mu(1) = 1$.*

Proof. Apply Lemma 5.2 in the case $\zeta = 1 \in \mathcal{R}$. Any measure $\mu : \mathcal{R} \rightarrow [0, 1] \subseteq \mathbb{R}$ for which $\mu(1) = 1$ takes addition and subtraction in \mathcal{R} to the usual addition and subtraction in \mathbb{R} . Consequently the formal continued fraction expansion of the formal quotient $\eta/1$ must be the same as the ordinary continued fraction expansion of $\mu(\eta)$. The measure μ is therefore completely determined by the structure of \mathcal{R} . ■

The theorem says we can identify any divided Archimedean scale with some subset of $[0, 1] \subseteq \mathbb{R}$.

DEFINITION 5.5. For ζ, θ in a linearly ordered divided Archimedean scale, the relation

$$\zeta \theta = \eta$$

means

$$\mu(\zeta)\mu(\eta) = \mu(\theta),$$

or equivalently the not-everywhere-defined multiplication is given by

$$\zeta \eta = \mu^{-1}(\mu(\zeta)\mu(\eta)).$$

EXAMPLE 5.9. Let ρ be the probability measure on the set of all subsets of $\{a, b, c\}$ that assigns $1/3$ to each of $\{a\}, \{b\}, \{c\}$. Then Definition 5.5 fails to define $\rho\{a\}\rho\{b\}$.

In §8.6 we will apply Definition 5.5 to probability.

6. HOMOMORPHISMS AND STONE SPACES

The material in this section is not new. All or nearly all of it can be found in [6].

6.1. Homomorphisms

DEFINITION 6.1.

1. Let \mathbb{A}, \mathbb{B} be Boolean algebras. A **homomorphism** $\varphi : \mathbb{A} \rightarrow \mathbb{B}$ is a function for which, for all $x, y \in \mathbb{A}$ we have:

$$\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$$

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$$

$$\varphi(\sim x) = \sim \varphi(x).$$

2. The **kernel** of a homomorphism $\varphi : \mathbb{A} \rightarrow \mathbb{B}$ is $\varphi^{-1}(0) = \{x \in \mathbb{A} : \varphi(x) = 0\}$.

3. A **principal** homomorphism is one whose kernel is of the form $\{y \in \mathbb{A} : y \leq x\}$ for some $x \in \mathbb{A}$. We say that the kernel is generated by x . Other homomorphisms are **nonprincipal** homomorphisms.

4. A homomorphism $\varphi : \mathbb{A} \rightarrow \mathbb{B}$ is **2-valued** if \mathbb{B} is the two-element Boolean algebra $\{0, 1\}$.

The next proposition is an immediate corollary of Definition 6.1.

PROPOSITION 6.1. *A 2-valued homomorphism φ on \mathbb{A} is principal if and only if for some atom $x \in \mathbb{A}$, $\varphi(y) = 1$ or 0 according as $x \leq y$ or $x \wedge y = 0$.*

EXAMPLE 6.1. Every finite Boolean algebra \mathbb{A} is isomorphic to the Boolean algebra of all subsets of some finite set Φ . Let \mathbb{B} be the Boolean algebra of all subsets of some non-empty set $\Psi \subseteq \Phi$. For $x \in \mathbb{A}$, let $\varphi(x) = x \cap \Psi$. Then φ is a principal homomorphism whose kernel is generated by $\Phi \setminus \Psi$. If Ψ is a single-element set, then φ is a 2-valued homomorphism.

EXAMPLE 6.2. Let \mathbb{A} be the Boolean algebra of all subsets of $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Let \mathbb{B} be the Boolean algebra of all equivalence classes of such sets, two sets $A, B \subseteq \mathbb{N}$ being considered equivalent if $|(A \setminus B) \cup (B \setminus A)| < \aleph_0$ (this is a much coarser equivalence relation than the one considered in Example 3.4!). Meet, join, and complement on \mathbb{B} are defined by choosing members of equivalence classes, then evaluating the meet, join, or complement of those, then taking the equivalence class to which the result belongs. It is easy to check that these operations are well-defined. For $x \in \mathbb{A}$, let $\varphi(x)$ be the equivalence class to which x belongs. This is a nonprincipal homomorphism whose kernel is the set of all finite subsets of \mathbb{N} .

EXAMPLE 6.3. Let \mathbb{A} be the Boolean algebra of all subsets of \mathbb{N} that are either finite or cofinite (cofinite means having a finite complement). For any $n \in \mathbb{N}$ the mapping

$$x \mapsto \left\{ \begin{array}{l} 1 \text{ if } n \in x \\ 0 \text{ if } n \notin x \end{array} \right\}$$

is a principal 2-valued homomorphism whose kernel is the set of all subsets of $\mathbb{N} \setminus \{n\}$. The mapping

$$x \mapsto \left\{ \begin{array}{l} 1 \text{ if } x \text{ is cofinite} \\ 0 \text{ if } x \text{ is finite} \end{array} \right\}$$

is a nonprincipal 2-valued homomorphism whose kernel is the set of all finite subsets of \mathbb{N} .

EXAMPLE 6.4. Let \mathbb{A} be the Boolean algebra of all clopen (i.e., simultaneously closed and open) subsets of the Cantor set C . Let \mathbb{B} be the Boolean algebra of all clopen subsets of $C \setminus [0, 1/3)$. The mapping $x \mapsto x \setminus [0, 1/3)$ is a nonprincipal homomorphism whose kernel is $\{x \in \mathbb{A} : x \subseteq [0, 1/3)\}$. Now fix one point $r \in C$. The mapping

$$x \mapsto \left\{ \begin{array}{l} 1 \text{ if } r \in x \\ 0 \text{ if } r \notin x \end{array} \right\} \tag{11}$$

is a nonprincipal 2-valued homomorphism whose kernel is $\{x \in \mathbb{A} : r \notin x\}$.

6.2. Stone's representation of Boolean algebras

In both Example 6.1 and Example 6.4, we saw a 2-valued homomorphism on a Boolean algebra \mathbb{A} of sets defined as in (11) – its value is 1 or 0 according as the set does or does not contain a certain point. In Example 6.1, the homomorphism was principal because the set containing only that one point was a member of \mathbb{A} ; in Example 6.4, it was nonprincipal because the set containing only that one point was not a member of \mathbb{A} . We shall see that in a sense, these examples are typical of 2-valued homomorphisms: We can represent an arbitrary Boolean algebra \mathbb{A} as the Boolean algebra of certain subsets of a certain set $\Phi(\mathbb{A})$, and then find that every 2-valued homomorphism is of the form (11). The homomorphism will be principal or nonprincipal according as the set containing only the point that so represents it is or is not one of the subsets of $\Phi(\mathbb{A})$ that are identified with members of the Boolean algebra \mathbb{A} .

So we let

$$\Phi(\mathbb{A}) = \text{the set of all 2-valued homomorphisms on } \mathbb{A},$$

and we identify each $x \in \mathbb{A}$ with

$$\{\varphi \in \Phi(\mathbb{A}) : \varphi(x) = 1\} \tag{12}$$

= the set of all 2-valued homomorphisms on \mathbb{A} that map x to 1. The operations of meet, join, and complement in \mathbb{A} correspond to the operations of (finite) intersection, (finite) union, and set-theoretic complementation on subsets of $\Phi(\mathbb{A})$. (Infinitary operations are more problematic. The infinitary join of $\mathbb{X} \subseteq \mathbb{A}$ is the smallest upper bound $\bigvee \mathbb{X}$ of \mathbb{X} in \mathbb{A} . This join does not always exist – counterexamples can be found within Example 6.4. When the join does exist, it does not generally correspond to the union of $\bigcup_{x \in \mathbb{X}} \{\varphi \in \Phi(\mathbb{A}) : \varphi(x) = 1\}$, since the union of sets of the form (12) is not generally of the form (12). Rather, the join corresponds to the smallest set of the form (12) that includes the union.)

The next result is *Stone's representation theorem*.

THEOREM 6.1. *The mapping*

$$x \mapsto \Phi(x) = \{\varphi \in \Phi(\mathbb{A}) : \varphi(x) = 1\}$$

is an isomorphism from the Boolean algebra \mathbb{A} , to the Boolean algebra of sets of the form (12) with the operations of intersection, union, and set-theoretic complementation in the roles of meet, join, and complement.

Proof. First we show that Φ is a homomorphism.

$$\begin{aligned}\Phi(\sim x) &= \{ \varphi \in \Phi(\mathbb{A}) : \varphi(\sim x) = 1 \} \\ &= \{ \varphi \in \Phi(\mathbb{A}) : \varphi(x) \neq 1 \} \text{ since } \varphi \text{ is a 2-valued homomorphism,} \\ &= \Phi(\mathbb{A}) \setminus \{ \varphi \in \Phi : \varphi(x) = 1 \} \\ &= \Phi(\mathbb{A}) \setminus \Phi(x).\end{aligned}$$

So complements in \mathbb{A} go to set-theoretic complements.

$$\begin{aligned}\Phi(x \wedge y) &= \{ \varphi \in \Phi(\mathbb{A}) : \varphi(x \wedge y) = 1 \} \\ &= \{ \varphi \in \Phi(\mathbb{A}) : \varphi(x) \wedge \varphi(y) = 1 \} \text{ since } \varphi \text{ is a homomorphism,} \\ &= \{ \varphi \in \Phi(\mathbb{A}) : \varphi(x) = 1 \text{ and } \varphi(y) = 1 \} \\ &= \{ \varphi \in \Phi(\mathbb{A}) : \varphi(x) = 1 \} \cap \{ \varphi \in \Phi(\mathbb{A}) : \varphi(y) = 1 \} \\ &= \Phi(x) \cap \Phi(y).\end{aligned}$$

So meets in \mathbb{A} go to intersections.

Let “ \vee ”, “or”, and “ \cup ” replace “ \wedge ”, “and”, and “ \cap ” respectively, to show that joins in \mathbb{A} go to unions. So Φ is indeed a homomorphism.

To show that it is an isomorphism, we need to show that it is one-to-one. For $x, y \in \mathbb{A}$ let

$$x + y = (x \wedge \sim y) \vee (y \wedge \sim x) = (x \vee y) \wedge \sim(x \wedge y),$$

and let

$$xy = x \wedge y.$$

Then it can be checked that \mathbb{A} becomes a commutative ring with zero element 0 and unit element 1, in which every element is idempotent and every element is its own additive inverse. The Boolean operations of meet, join, and complement can be recovered from the ring operations:

$$\begin{aligned}x \wedge y &= xy, \\ x \vee y &= x + y + xy, \\ \sim x &= 1 + x.\end{aligned}$$

And Boolean homomorphisms coincide exactly with ring homomorphisms. The kernel $\{ x \in \mathbb{A} : \Phi(x) = \emptyset \}$ of the Boolean homomorphism is the same thing as the kernel of the ring homomorphism. Therefore, to show that Φ is one-to-one, it is enough to show that the kernel contains only $0 \in \mathbb{A}$. That is the same as showing that if $x > 0$ then $\Phi(x) \neq \emptyset$. In other words,

if $x \neq 0$ then for some 2-valued homomorphism φ on \mathbb{A} we have $\varphi(x) = 1$. Equivalently, if $x \neq 1$ then for some 2-valued homomorphism φ on \mathbb{A} we have $\varphi(x) = 0$. The kernel of such a homomorphism is a proper ideal. The fact that it has only two cosets implies that it is a *maximal* proper ideal. So we need only show that any $x \neq 1$ is a member of some maximal proper ideal. That is well-known to follow from a standard application of Zorn's lemma. ■

6.3. Topology

As in Example 6.4 these sets $\{\varphi \in \Phi : \varphi(x) = 1\}$ will be the clopen subsets of Φ – but to say that, we need a topology on Φ . Here it is.

DEFINITION 6.2. The **Stone space** $\Phi(\mathbb{A})$ of a Boolean algebra \mathbb{A} is the set Φ of all 2-valued homomorphisms on \mathbb{A} endowed with the topology whose basic open sets are sets of the form (12). That means the open sets are just those that are unions of arbitrary collections of sets of the form (12).

This is the same as the topology of pointwise convergence of nets of homomorphisms. That the basic open sets are closed follows immediately from the fact that the basic open set $\Phi(\sim x)$ is complementary to the basic open set $\Phi(x)$. That the basic open sets are the *only* clopen sets is proved in [6] by using the following theorem. But our real motive for including this theorem is its use in §§7 and 8.

THEOREM 6.2. (*Compactness*) Let $\mathbb{X} \subseteq \mathbb{A}$. Suppose for every finite subset $\mathbb{X}_0 \subseteq \mathbb{X}$ there is a 2-valued homomorphism φ on \mathbb{A} such that for every $x \in \mathbb{X}_0$, $\varphi(x) = 1$. Then there is a 2-valued homomorphism φ on \mathbb{A} such that for every $x \in \mathbb{X}$, $\varphi(x) = 1$.

In other words, the Stone space is compact.

Proof. We follow closely the argument in [6] pp. 77-78. It suffices to prove $\Phi(\mathbb{A})$ is a closed subset of the space Ω of *all* functions (not just homomorphisms) from \mathbb{A} into $\{0, 1\}$, with the product topology, since that

is a compact Hausdorff space. We have

$$\begin{aligned} \Phi(\mathbb{A}) &= \left(\bigcap_{x \in \mathbb{A}} \{ \varphi \in \Omega : \varphi(\sim x) = \sim \varphi(x) \} \right) \\ &\cap \left(\bigcap_{x, y \in \mathbb{A}} \{ \varphi \in \Omega : \varphi(x \vee y) = \varphi(x) \vee \varphi(y) \} \right) \\ &\cap \left(\bigcap_{x, y \in \mathbb{A}} \{ \varphi \in \Omega : \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) \} \right). \end{aligned}$$

This is closed if the sets whose intersection is taken are closed. They are closed because $\varphi(x)$ depends continuously on φ . ■

EXAMPLE 6.5. If \mathbb{A} is the Boolean algebra of all finite or cofinite subsets of \mathbb{N} , then $\Phi(\mathbb{A})$ is the one-point compactification of the discrete space whose underlying set is \mathbb{N} . The isolated points of $\Phi(\mathbb{A})$ correspond to principal 2-valued homomorphisms. The one limit point corresponds to the one nonprincipal 2-valued homomorphism, which maps cofinite sets to 1 and finite sets to 0.

EXAMPLE 6.6. If \mathbb{A} is the Boolean algebra of all subsets of \mathbb{N} , then $\Phi(\mathbb{A})$ is the Stone-Cech compactification of the discrete space whose underlying set is \mathbb{N} . Again, the isolated points correspond to the principal 2-valued homomorphisms, and the $2^{2^{\aleph_0}}$ limit points to the nonprincipal 2-valued homomorphisms. If \mathbb{B} is the Boolean algebra of equivalence classes of such sets, where two sets are equivalent if and only if their symmetric difference is finite, then the Stone space $\Phi(\mathbb{B})$ of this atomless Boolean algebra is the set of all limit points of $\Phi(\mathbb{A})$.

EXAMPLE 6.7. If \mathbb{A} is, as in Example 6.4, the Boolean algebra of all clopen subsets of the Cantor set C , then $\Phi(\mathbb{A}) = C$.

7. CONTINUITY OF SCALINGS

7.1. Definition and examples

DEFINITION 7.1. A scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is **continuous** at a homomorphism $\varphi : \mathbb{A} \rightarrow \mathbb{B}$ if

$$\rho \left(\bigwedge \{ x : \varphi(x) = 1 \} \right) = \bigwedge \{ \rho(x) : \varphi(x) = 1 \} \quad (13)$$

or, equivalently

$$\rho\left(\bigvee\{x:\varphi(x)=0\}\right)=\bigvee\{\rho(x):\varphi(x)=0\}. \quad (14)$$

We shall see that continuity at every principal homomorphism is like “continuity of measure,” and continuity at every 2-valued homomorphism at least sometimes entails “Archimedeanism.”

EXAMPLE 7.1. If \mathbb{A} is finite, then every scaling on \mathbb{A} is continuous at every homomorphism on \mathbb{A} .

EXAMPLE 7.2. Let φ be the canonical homomorphism from the Boolean algebra of all subsets of \mathbb{N} into quotient algebra of that Boolean algebra by the ideal of finite subsets of \mathbb{N} . In other words, for $A \subseteq \mathbb{N}$ we have $\varphi(A) = 0$ if and only if A is finite, or, equivalently, for $A, B \subseteq \mathbb{N}$ we have $\varphi(A) = \varphi(B)$ if and only if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ is finite. Let ρ be the “simple non-Archimedean scaling” of Example 3.4. Then ρ is discontinuous at φ . To see this, observe that

$$\begin{aligned} \rho\left(\bigwedge\{\{n, n+1, n+2, \dots\} : n \in \mathbb{N}\}\right) &= \rho(\emptyset) \\ &< \rho(\emptyset) + 5 \leq \rho(\{n, n+1, n+2, \dots\}) \quad \text{for every } n \in \mathbb{N}. \end{aligned}$$

EXAMPLE 7.3. (A completely additive measure) Let \mathbb{A} be the quotient algebra of Lebesgue-measurable subsets of the interval $[0, 1]$ on the real line by the ideal of sets of measure 0. Let ρ be the quotient measure of Lebesgue measure on \mathbb{A} . This scaling is continuous at all principal homomorphisms on \mathbb{A} .

7.2. Continuity and additivity

DEFINITION 7.2. A subset \mathbb{X} of a Boolean algebra \mathbb{A} is **pairwise disjoint** if any distinct $x, y \in \mathbb{X}$ are disjoint, i.e., for any $x, y \in \mathbb{X}$, if $x \neq y$ then $x \wedge y = 0$.

DEFINITION 7.3. Suppose $\mathbb{X} \subseteq \mathbb{A}$ is pairwise disjoint. Then the sum on the left side of the equality below is defined to be the join on the right. The sum exists whenever the join exists.

$$\sum_{x \in \mathbb{X}} \rho(x) = \bigvee \left\{ \sum_{x \in \mathbb{X}_0} \rho(x) : \mathbb{X}_0 \text{ is a finite subset of } \mathbb{X} \right\}.$$

DEFINITION 7.4. A scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is **completely additive** if for every pairwise disjoint $\mathbb{X} \subseteq \mathbb{A}$ possessing a join $\bigvee \mathbb{X} \in \mathbb{A}$, we have

$$\sum_{x \in \mathbb{X}} \rho(x) = \rho\left(\bigvee \mathbb{X}\right).$$

The scaling of Example 7.3 is a completely additive measure. Lebesgue measure itself is only countably, and not completely, additive. In this example, sets of measure zero all belong to the same equivalence class, which is the zero-element of the quotient algebra. Consequently we cannot have any uncountable antichain (an “antichain” is pairwise disjoint collection of members of a poset) whose join is 1. Only such a collection could serve as the needed counterexample to complete additivity.

EXAMPLE 7.4. (A completely additive scaling on a Boolean algebra that does not satisfy the countable antichain condition) Let ρ be the identity mapping on the Boolean algebra of all subsets of the real line. Clearly ρ is completely additive. This Boolean algebra has uncountable antichains, i.e., it does not satisfy the “countable antichain condition.”

THEOREM 7.1. *Suppose a scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is continuous at every principal homomorphism on \mathbb{A} . Then ρ is completely additive.*

Proof. The problem is to show that if $\mathbb{X} \subseteq \mathbb{A}$ is pairwise disjoint and has a join in \mathbb{A} then

$$\rho\left(\bigvee \mathbb{X}\right) = \sum_{x \in \mathbb{X}} \rho(x) = \bigvee \left\{ \sum_{x \in \mathbb{X}_0} \rho(x) : \mathbb{X}_0 \text{ is a finite subset of } \mathbb{X} \right\}.$$

Observe that

$$\bigvee \mathbb{X} = \bigvee \left\{ \bigvee \mathbb{X}_0 : \mathbb{X}_0 \text{ is a finite subset of } \mathbb{X} \right\}.$$

Therefore by condition (14), characterizing continuity, and the assumption that ρ is continuous at every principal homomorphism, it suffices that there be a principal homomorphism whose kernel is $[0, \bigvee \mathbb{X}]$. That homomorphism is $x \mapsto x \wedge \sim \bigvee \mathbb{X}$ from \mathbb{A} into the relative Boolean algebra $[0, \sim \bigvee \mathbb{X}]$. ■

7.3. Continuity and Archimedeanism

DEFINITION 7.5. A member $\delta \in \mathcal{R}$ is an **infinitesimal** for a basic scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ if for some infinite pairwise disjoint $\mathbb{X} \subseteq \mathbb{A}$ we have $\rho(x) \geq \delta$ for every $x \in \mathbb{X}$.

EXAMPLE 7.5. The zero element of any scale is an infinitesimal.

EXAMPLE 7.6. If $\rho : \mathbb{A} \rightarrow \mathcal{R} = [0, 1] \subseteq \mathbb{R}$ is a measure, then there are no nonzero infinitesimals in \mathcal{R} .

EXAMPLE 7.7. In Example 3.4, every member of the “initial galaxy”

$$\{\rho(\emptyset), \rho(\emptyset) + 1, \rho(\emptyset) + 2, \rho(\emptyset) + 3, \dots\}$$

is an infinitesimal.

EXAMPLE 7.8. The identity mapping from any Boolean algebra to itself is a basic scaling; the algebra regarded as a scale has no infinitesimals.

PROPOSITION 7.1. *Suppose $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is a basic scaling and $\sigma : \mathcal{R} \rightarrow \mathcal{S}$ is a scaling. (Recall that according to Definition 2.6, σ “extends” ρ .) If $\delta \in \mathcal{R}$ is an infinitesimal, then so is $\sigma(\delta) \in \mathcal{S}$.*

It is easy to see that the converse is false:

EXAMPLE 7.9. The Boolean algebra of all subsets of any set, viewed as a scale, contains no infinitesimals.

In other words, extending a scale can create infinitesimals but cannot destroy them.

So now we have motivated the next definition.

DEFINITION 7.6.

1. Let $\rho : \mathbb{A} \rightarrow \mathcal{R}$ be a divided basic scaling. The scale $\mathcal{R} = \{\rho(x) : x \in \mathbb{A}\}$ is **Archimedean** if it has no nonzero infinitesimals, and **non-Archimedean** if it contains at least one nonzero infinitesimal.

2. An Archimedean divided scale is **stably Archimedean** if there is no scaling $\sigma : \mathcal{R} \rightarrow \mathcal{S}$ extending the basic scaling ρ , such that \mathcal{S} contains any nonzero infinitesimal, and **unstably Archimedean** if it is Archimedean but not stably Archimedean.

The term “stably Archimedean” was suggested by Timothy Chow and Daniel Lueking independently of each other, in response to a request for suggested nomenclature posted to the usenet newsgroup `sci.math.research`.

Why does Definition 7.6 say “divided”? Suppose $\rho(x) = \alpha < \beta = \rho(y)$, and there is some infinite pairwise disjoint collection \mathbb{U} of members of the domain of ρ such that for any $u \in \mathbb{U}$ we have $u \wedge x = 0$ and $\rho(x) + \rho(u) \geq \rho(y)$. Divideneess implies we can subtract α from β , and Definition 7.5 then implies $\beta - \alpha$ is an infinitesimal. Without dividedness I see no way to guarantee that any nonzero lower bound of $\{\rho(u) : u \in \mathbb{U}\}$ exists. Thus,

without divideness, it is conceivable that two members of a scale could differ infinitesimally, even though no member differs infinitesimally from 0. I am indebted to an anonymous referee for this point. If I knew any such example, I would consider emending Definition 7.6. (The referee speculated that if F is an ordered field of which the real field \mathbb{R} is a subfield, so that F contains infinitesimals, then the set

$$\{ \alpha \in F : 0 \leq \alpha \leq 1 \text{ and neither } \alpha \text{ nor } 1 - \alpha \text{ is a nonzero infinitesimal} \}$$

would be such a case. But it must be remembered that, by our definitions, the addition on a scale is inherited from a scaling whose domain is some Boolean algebra. No such mapping was proposed.)

THEOREM 7.2. *Suppose a scaling $\rho : \mathbb{A} \rightarrow \mathcal{R}$ is continuous at every 2-valued homomorphism on \mathbb{A} , and \mathcal{R} is linearly ordered. Then \mathcal{R} has no infinitesimals. (Consequently, if \mathcal{R} is divided, it is Archimedean.)*

Proof. Suppose $\delta > 0$ is an infinitesimal in \mathcal{R} . We have seen that for any 2-valued homomorphism φ , the infimum $\bigwedge \{ x : \varphi(x) = 1 \}$ exists, and

$$\bigwedge \{ x : \varphi(x) = 1 \} = \begin{cases} \text{an atom } x_\varphi & \text{if } \varphi \text{ is principal} \\ 0 & \text{if } \varphi \text{ is nonprincipal} \end{cases}.$$

In the principal case, for each $x \in \mathbb{A}$ we have

$$\varphi(x) = \begin{cases} 1 & \text{if } x_\varphi \leq x \\ 0 & \text{if } x_\varphi \not\leq x \end{cases}.$$

We have now defined x_φ when φ is a principal homomorphism; next we shall define x_φ in terms of δ when φ is a nonprincipal homomorphism. In the latter case, since the greatest lower bound $\bigwedge \{ \rho(x) : \varphi(x) = 1 \}$ is $0 < \delta$, it must be that δ is not a lower bound, and that means some $x \in \mathbb{A}$ satisfies $\varphi(x) = 1$ and $\rho(x) < \delta$. Choose such an x and call it x_φ . Via Stone's duality we can identify x_φ with a clopen subset of the Stone space — the set of all 2-valued homomorphisms that map x_φ to 1 — which contains the point φ . Now we have a clopen cover $\{ x_\varphi : \varphi \in \Phi \}$ of the Stone space. Since the Stone space is compact, this has a finite subset $\{ x_{\varphi_1}, \dots, x_{\varphi_n} \}$ that covers the whole Stone space, so that $x_{\varphi_1} \vee \dots \vee x_{\varphi_n} = 1$. Some terms in this join — call them $x_{\varphi_{m+1}}, \dots, x_{\varphi_n}$ — may be atoms whose images under ρ are $\geq \delta$. The join $x_{\varphi_1} \vee \dots \vee x_{\varphi_m}$ of the others must be \geq any $x \in \mathbb{A}$ whose image under ρ is $< \delta$. Since these x 's need not be disjoint, we replace them with y_1, \dots, y_m such that $y_i \leq x_{\varphi_i}$ for $i = 1, \dots, m$,

$y_i \wedge y_j = 0$ for $i, j = 1, \dots, m$, and $y_1 \vee \dots \vee y_m = x_{\varphi_1} \vee \dots \vee x_{\varphi_m}$. (This can be done by letting $y_i = x_{\varphi_i} \wedge \sim(\dots \vee x_{\varphi_{i-1}})$ for each i .)

That δ is an infinitesimal means there is an infinite pairwise disjoint set $\mathbb{Z} \subseteq \mathbb{A}$ such that for each $z \in \mathbb{Z}$ we have $\rho(z) \geq \delta$. Since the complement of $y_1 \vee \dots \vee y_m$ consists of only finitely many atoms, no generality is lost by assuming

$$\bigvee_{z \in \mathbb{Z}} z \leq y_1 \vee \dots \vee y_m.$$

This inequality entails

$$\rho\left(\bigvee_{z \in \mathbb{Z}} z\right) \leq \rho(y_1 \vee \dots \vee y_m),$$

and that in turn entails the middle inequality below:

$$\sum_{z \in \mathbb{Z}} \delta \leq \sum_{z \in \mathbb{Z}} \rho(z) \leq \sum_{i=1}^m \rho(y_i) \leq \underbrace{\delta + \dots + \delta}_{m \text{ terms}}.$$

Since the first sum has infinitely many terms and $\delta > 0$, that is not consistent with Lemma 4.3 (4). \blacksquare

THEOREM 7.3. *If a scale is linearly ordered, Archimedean, and divided, then it is measurable.*

Proof. For any $\alpha, \beta \in \mathcal{R}$, linear ordering implies that either $\alpha < \sim \beta$, $\alpha > \sim \beta$, or $\alpha = \sim \beta$. Dividedness then entails that in the first case, $\alpha + \beta$ exists and $\alpha \oplus \beta$ does not, in the second case $\alpha \oplus \beta$ exists and $\alpha + \beta$ does not, and in the third case they both exist, and $\alpha + \beta = 1$ and $\alpha \oplus \beta = 0$.

We define an abelian group G whose underlying set is $\mathbb{Z} \times \mathcal{R}$, i.e., the set of all ordered pairs (n, α) where n is an integer and $\alpha \in \mathcal{R}$, modulo the identification of $(n+1, 0)$ with $(n, 1)$, for each $n \in \mathbb{Z}$. The addition in this group is

$$(n, \alpha) + (m, \beta) = \begin{cases} (n+m, \alpha + \beta) & \text{if } \alpha \leq \sim \beta \\ (n+m+1, \alpha \oplus \beta) & \text{if } \alpha \geq \sim \beta \end{cases} \quad (15)$$

The identification of $(n+1, 0)$ with $(n, 1)$ keeps the two pieces of this definition from contradicting each other. We linearly order this group by saying that if $\alpha, \beta \neq 1$ then $(n, \alpha) < (m, \beta)$ if either $n < m$, or $n = m$ and $\alpha < \beta$. This linear ordering is compatible with the group operation, in the sense that for any $0 \neq u \in G$, either $u > 0$ or $-u > 0$, and for any

$u, v, w \in G$, if $u < v$ then $u + w < v + w$. A group with such a compatible linear ordering is a “linearly ordered group.”

Observe that the Archimedean nature of \mathcal{R} and that of \mathbb{Z} together imply that G is Archimedean in the sense that for any $u, v > 0$ in G , there is some positive integer n such that

$$\underbrace{u + \cdots + u}_n > v,$$

so that no matter how small u is by comparison to v , it takes only finitely many u 's to add up to more than v .

A well-known theorem of Hölder (see [5], p. 45) says that if a linearly ordered group G is Archimedean, then there is an isomorphism f from G into the additive group of real numbers. For $\alpha \in \mathcal{R}$, so that $(0, \alpha) \in G$, let $\mu(\alpha) = f(0, \alpha)$. Then μ is the desired measure. ■

Note that any extension of Example 5.7 to a linearly ordered scale is a counterexample showing that the hypothesis of divisibility cannot be dispensed with.

I do not know how to prove the following.

CONJECTURE. The hypothesis of linear ordering in Theorem 7.3 can be dropped.

8. DEGREES OF BELIEF

8.1. Boolean algebra models propositional logic

Propositional logic studies finitary logical connectives like “and”, “or”, “not”, which connect propositions.

Every proposition is either true or false. Suppose some are *known* to be true, some are *known* to be false, and the truth values of some others are uncertain. Call two propositions x and y (conditionally) equivalent (given what is known) if the proposition [x if and only if y] is known to be true. It is easy to check that if x_1 is equivalent to x_2 and y_1 is equivalent to y_2 then [x_1 and y_1] is equivalent to [x_2 and y_2], [x_1 or y_1] is equivalent to [x_2 or y_2], and [not x_1] is equivalent to [not x_2]. Therefore we can think of the three connectives “and”, “or”, “not” as acting on equivalence classes rather than on propositions.

Any set of such equivalence classes of propositions that is closed under these three connectives necessarily contains the equivalence class, which we shall call 1, of propositions known to be true, and the class, which we shall call 0, of propositions known to be false. If the truth values of an

(equivalence class of) proposition(s) is uncertain, then the set in question also contains other classes than 0 and 1. That set of equivalence classes of propositions then constitutes a Boolean algebra with the connectives “and”, “or”, and “not” in the roles of meet, join, and complement. The natural partial order of this Boolean algebra makes $x \leq y$ precisely if the proposition [if x then y] is known to be true.

8.2. Intrinsic possibility versus epistemic possibility

Possibility, like probability, can be either intrinsic or epistemic. To say it is possible that a card chosen randomly from a deck will be an ace, could be taken to mean that at least one ace is in the deck. That is *intrinsic* possibility. To say it is possible that the card that was drawn yesterday was an ace, could be taken to mean, not that some aces are in the deck, but that it is not certain that none are. That is *epistemic* possibility.

EXAMPLE 8.1. Following the notation of §8.1, we can say that $x < y$ means it is possible that y is true and x is false, but it is necessary that y is true if x is true. I was asked whether “it is possible that y is true and x is false” means

1. It is known that y is possible without x ; or
2. It is not known that y is impossible without x .

The punch line: If possibility is regarded as intrinsic, then (1) differs in meaning from (2), but if possibility is regarded as epistemic, then there is no difference!

Henceforth we regard possibility as epistemic, not intrinsic. That means, in particular, that we shall not speak of x as “occurring” or “not occurring,” but rather, as we did earlier, of x as being true or false, or as being known to be true, known to be false, or uncertain.

Notice that the notation

$$0 \leq x < y \leq 1$$

can be thought of as saying x is closer to being *known* to be false than y is, or y is closer to being *known* to be true than x is. Consequently we put a greater degree of belief in the truth of y than in that of x .

8.3. Some axioms of epistemic probability

The last paragraph of the last section hints at an axiom for epistemic probability theory: If x is less (epistemically) possible than y , then x is less (epistemically) probable than y . In other words, for any assignment P of

probabilities to propositions

$$\text{If } x < y \text{ then } P(x) < P(y). \tag{16}$$

We shall also take it to be axiomatic that the less probable x is, the more probable [not x] is, i.e.,

$$\text{If } P(x) < P(y) \text{ then } P(\text{not } x) > P(\text{not } y). \tag{17}$$

A third axiom is very similar to the “sure-thing principle” stated by Leonard Jimmie Savage in [11], pp. 21-2. It says that if x is no more probable than y given that z is true, and x is no more probable than y given that z is false, then x is no more probable than y given no information about whether z is true or false. In other words

$$\text{If } P(x | z) \leq P(y | z) \text{ and } P(x | \text{not } z) \leq P(y | \text{not } z) \text{ then } P(x) \leq P(y) \tag{18}$$

and “ $<$ ” holds in the consequent if it holds in either of the two antecedents. (Savage’s “sure-thing principle” spoke of utilities rather than of probabilities.) We do not understand an expression like “ $P(\bullet | \bullet)$ ” to mean anything different from something like “ $P(\bullet)$ ”; we take all probabilities to be conditional on some corpus of knowledge. So in particular, (17) implies that if $P(x | z) \leq P(y | z)$ then $P(\text{not } x | z) \geq P(\text{not } y | z)$.

It is unfortunate that, as things now stand, we must rely on one more assumption about degrees of belief in uncertain propositions – that they are linearly ordered:

$$\text{For all } x, y \text{ either } P(x) \leq P(y) \text{ or } P(y) \leq P(x). \tag{19}$$

This means we will have the conclusion we want for linearly ordered scales and for scales that are Boolean algebras – the two extreme cases – but not for intermediate cases.

Clearly (16) says that assignments of probabilities to propositions must satisfy part 1(i) of Definition 2.3. But (17) is weaker than part 1(ii) of Definition 2.3. If we can show that (17), (18), and (19) require assignments of probabilities to propositions to satisfy part 1(ii) of Definition 2.3 then we will know that all such assignments must be basic scalings. That is what we do in the next section.

8.4. Linearly ordered scales as probability assignments

We want to show that (17), (18), and (19) require assignments of probabilities to propositions to satisfy part 1(ii) of Definition 2.3. Part 1(ii) of

Definition 2.3 speaks of relative complementation. In propositional logic, relative complementation is relative logical negation. If $a \leq x \leq b$, meaning a is a sufficient condition for x , and b is a necessary condition for x , then the logical negation of x relative to the interval $[a, b]$ is the unique (up to logical equivalence) proposition u such that

1. a is a sufficient condition for u , and
2. b is a necessary condition for u , and
3. u becomes equivalent to $[\text{not } x]$ once it is learned that b is true and a is false.

That proposition is $[a \text{ or } (b \text{ and not } x)]$, or, equivalently (since a logically entails b) $[b \text{ and } (a \text{ or not } x)]$. So the problem is to show that (17), (18), and (19) imply that if

$$\begin{aligned} a &\leq x \leq b, \\ a &\leq y \leq b, \end{aligned}$$

and $P(x) \leq P(y)$, then $P(a \text{ or } [b \text{ and not } x]) \geq P(a \text{ or } [b \text{ and not } y])$.

If, to get a contradiction, we assume on the contrary that

$$P(a \text{ or } [b \text{ and not } x]) \not\geq P(a \text{ or } [b \text{ and not } y])$$

then (19) tells us that

$$P(a \text{ or } [b \text{ and not } x]) < P(a \text{ or } [b \text{ and not } y]).$$

The conjunction of this inequality with (18) means we cannot have both

$$P(a \text{ or } [b \text{ and not } x] \mid b \text{ and not } a) \geq P(a \text{ or } [b \text{ and not } y] \mid b \text{ and not } a) \quad (20)$$

and

$$\begin{aligned} &P(a \text{ or } [b \text{ and not } x] \mid \underline{\text{not}} \{b \text{ and not } a\}) \\ &= P(a \text{ or } [b \text{ and not } y] \mid \underline{\text{not}} \{b \text{ and not } a\}). \end{aligned} \quad (21)$$

The equality in (21) is trivially true because the condition $[\text{not } (b \text{ and not } a)]$ renders impossible the propositions whose probability is being taken. Therefore (20) must be false. So, by (19), we must have

$$P(a \text{ or } [b \text{ and not } x] \mid b \text{ and not } a) < P(a \text{ or } [b \text{ and not } y] \mid b \text{ and not } a). \quad (22)$$

Given $[b \text{ and not } a]$, the propositions $[a \text{ or } (b \text{ and not } x)]$ and $[a \text{ or } (b \text{ and not } y)]$ simplify to $[\text{not } x]$ and $[\text{not } y]$ respectively, and (22) simplifies to

$$P(\text{not } x \mid b \text{ and not } a) < P(\text{not } y \mid b \text{ and not } a). \quad (23)$$

We must also have

$$P(\text{not } x \mid \underline{\text{not}} \{b \text{ and not } a\}) = P(\text{not } y \mid \underline{\text{not}} \{b \text{ and not } a\}) \quad (24)$$

because, given the condition $[\text{not } (b \text{ and not } a)]$, the two propositions $[\text{not } x]$ and $[\text{not } y]$ are equivalent to each other. Conjoining (23), (24), and (18), we conclude that $P(\text{not } x) < P(\text{not } y)$. In view of (17) and our assumption that $P(x) \leq P(y)$, this is impossible.

We conclude that assignments of linearly ordered probabilities to uncertain propositions should be scalings. As scalings, they must satisfy all of our results on addition, dual-addition, subtraction, relative complementation, modularity, and de-Morganism.

8.5. Finiteness of information-content in propositions

When would a scaling need to be continuous in order to model properly the phenomenon of assignment of degrees of belief to uncertain propositions?

Suppose a subset \mathbb{X} of some Boolean algebra \mathbb{A} of propositions is closed under “and” (i.e., $[x \text{ and } y] \in \mathbb{X}$ for any $x, y \in \mathbb{X}$) and satisfies $\bigwedge \mathbb{X} = 0$ and $x > 0$ for every $x \in \mathbb{X}$. The simplest example is the Boolean algebra \mathbb{A} that is freely generated by x_1, x_2, x_3, \dots , i.e., the set of all propositions constructed from x_1, x_2, x_3, \dots by using only finitely many occurrences of “and”, “or”, and “not”, and \mathbb{X} is the set $\{x_1, x_2, x_3, \dots\}$ of generators. If a probability α satisfies $0 < \alpha \leq P(x)$ for every $x \in \mathbb{X}$, then it would seem appropriate to consider α to be a probability assigned only to propositions that convey an amount of information that is infinite by comparison to that conveyed by any $x \in \mathbb{X}$. Closure of \mathbb{X} under “and” is the same as closure of a family of clopen subsets of the Stone space under finite intersections. Consequently, compactness of the Stone space implies that for some 2-valued homomorphism φ we have $\varphi(x) = 1$ for every $x \in \mathbb{X}$. The probability α described above would then be a counterexample to the continuity of P at φ . So exclusion from a scale, of probabilities assigned only to propositions that convey an infinite amount of information, amounts to continuity of the assignment of probabilities at every 2-valued homomorphism. In view of Theorems 7.2 and 7.3, then, an insistence on finite information content in propositions, implies that the scale on which the probabilities are measured

is measurable. In other words, we may take the probabilities to be real numbers, and the scale to be $[0, 1] \subseteq \mathbb{R}$.

8.6. Multiplication

From (18) it follows that

$$P(x \text{ and } z) \leq P(y \text{ and } z) \text{ if and only if } P(x | z) \leq P(y | z). \quad (25)$$

By symmetry the same is true if “ \geq ” replaces both occurrences of “ \leq ”, and consequently also of “ $=$ ” replaces both.

If, as in §5.6, we assume linear ordering, dividedness, and Archimdeanism, then there exists a maximum non-negative integer n_1 such that

$$P(z) - \underbrace{P(x \text{ and } z) - \dots - P(x \text{ and } z)}_{n_1} \quad (26)$$

exists, and the difference is less than $P(x \text{ and } z)$. Because of (25), this integer n_1 must be the same as the smallest non-negative integer n such that

$$1 - \underbrace{P(x | z) - \dots - P(x | z)}_n \quad (27)$$

exists. This latter difference must be less than $P(x | z)$. The process can be iterated according to the algorithm described in §5.6, and (25) tells us at each step that the entries $n_1, n_2, n_3 \dots$ in the formal continued fraction expansion of the formal quotient $P(x \text{ and } z)/P(z)$ are the same as those in the formal continued fraction expansion of the formal quotient $P(x | z)/1$. Thus we have:

THEOREM 8.1. *If probabilities are measured on a scale that is linearly ordered, divided, and Archimedean, then for any propositions x and z ,*

$$P(x \text{ and } z) = P(x | z)P(z).$$

ACKNOWLEDGMENTS

James M. Dickey, Karel Priky, and Gian-Carlo Rota all encouraged me in different ways to work on this topic. From a usenet posting by Włodzimierz Holsztynski I first heard of Kleene algebras, and he assisted in my understanding them in a subsequent exchange of e-mail. Richard Stanley’s comment on the confusing nature of a bit of

nomenclature inspired the term “additive relative complement.” Sara Billey suggested a need to clarify the “two-page crash course on nonstandard analysis” that appears in Example 3.5. Timothy Chow and Daniel Lueking independently suggested the term “stably Archimedean.”

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